Exponentials

1. Motivating Example

Suppose the population of monkeys on an island increases by 6% annually. Beginning with 455 animals in 2008, estimate the population for 2011 and for 2025.

Each year the number of monkeys increases by 6%. To determine the number of monkeys in 2011, begin by finding the number of monkeys in 2009.

Year		
2008	Population	455
	6% Increase	$0.06 \cdot 455 = 27.3$
2009	Population	482.3
	6% Increase	$0.06 \cdot 482.3 = 28.9$
2010	Population	511.2
	6% Increase	$0.06 \cdot 511.2 = 30.7$
2011	Population	541.9

Begin by multiplying the current population in 2008 by the percent of increase. Then add this result to the 2008 population to determine the 2009 population.

In reality, 0.3 or 0.9 of a monkey does not make much sense. Feel free to round to the nearest whole number for applications.

There were approximately 542 monkeys on the island in 2011. To find this value, only a few steps were needed; however, the process to find the population in 2025 would take much more work.

Making some observations:

2008: 455

2009: $455 + 0.06 \cdot 455 = 455(1 + 0.06) = 455(1.06)$

2010: $455(1.06) + 0.06 \cdot 455(1.06)$
$$= 455(1.06)(1 + 0.06) = 455(1.06)(1.06) = 455(1.06)^2$$

2011: $455(1.06)^2 + 0.06 \cdot 455(1.06)^2$
$$= 455(1.06)^2(1 + 0.06) = 455(1.06)^2(1.06) = 455(1.06)^3$$

Thus, the population from year to year is increased by multiplying the previous year's population by a factor of $M = 1.06$. One could then safely say that this process can be used to estimate the population 17 years later (2025) and whose value is

$$455(1.06)^{17} = 455(2.6928) \approx 1225$$

The island then has a population of 1225 monkeys in 2025.

2. Exponential Growth and Decay

The previous is an example of ***geometric growth***. Any quantity that increases by being multiplied by the same value on regular intervals is said to grow geometrically. In other words, each subsequent value is equal to the previous value multiplied by some constant multiplier, M.

In the previous scenario, the constant multiplier was $(1.06) = (1 + 0.06) = (1 + r)$. Letting $M = (1 + r)$ we can express a general geometric function as

$$y = y_0 M^x$$

where,

- y_0 is the initial condition (initial value)

- M is the multiplier, defined as $M = (1 + r)$ where r is the growth rate as a percent.

    - If the problem presented is that of ***decay*** (the value is getting smaller at a multiple rate), then $-1 \leq r < 0$.

    - Typically, $-1 \leq r \leq 1$. However it is possible for $|r| \geq 1$, especially in biology.
        - If $r > 0$ then you are referring to growth and
        - If $r < 0$ then you are referring to decay.

- x is the independent variable, often a measure of time.

The above is an example of ***discrete time***. Discrete time views values of variables as occurring at distinct, separate points in time. Here, a variable moves from one value to another as time moves from time period to the next. The number of measurements between any two time periods is finite. Measurements are typically made at sequential *integer* values of the time variable.

For instance, if calculating
- annual plant populations,
- the number of organisms with discrete life stages (e.g., juveniles or adults), or
- small population sizes (i.e., identifiable individuals)

then one would use discrete time models.

In contrast, ***continuous time*** views variables as having a particular value for potentially an infinitesimally short amount of time. Between any two points in time there are an infinite number of other points in time. The time variable ranges over the entire real number line, or depending on the context, over some subset of it such as the non-negative reals. The continuous time model is often referred to as ***exponential growth*** or ***exponential decay***.

Here, one would use continuous time models if trying to determine
- bacterial growth,
- the concentration of a chemical over the course of time, or
- large population sizes (i.e., the individual unit is not identifiable).

Ex.) A fish population increases from 357 to 411 fish within one year. Assume that the population keeps growing at the same rate, estimate the population five years later (5 years from the first count).

Here, $y_0 = 357$. After 1 year there are 411 fish, or when $x = 1, y = 411$.

$y = y_0 M^x$
$411 = 357M^{(1)}$
$\frac{411}{357} = M$
$1.151 = M$

$M = 1 + r \Rightarrow r = M - 1 = 1.151 - 1 = 0.151$

The question asks for the population 5 years later. The task is to find y when $x = 5$.
$y = y_0 M^x = 357 \cdot 1.151^5 = 722$

After 5 years, there are approximately 722 fish.

Ex.) **Infant Mortality Rate (**Number of infant deaths per 1,000 live births). The infant mortality in the United States was 8.81 in 1990 and it declined to 6.92 for 2000 (these are actually five year averages, source: United Nations).
Find an exponential model for the data, and estimate the infant mortality for 2010 and for 2025.

There is no rule that forces us to measure time in years. For this example decades are more appropriate. The initial value is 8.81. After 1 decade $(x = 1)$ the value is 6.92

$y = y_0 M^x$
$6.92 = 8.81M^{(1)}$
$\frac{6.92}{8.11} = M$
$0.7855 = M$

Thus, the mortality rate is declining by 21.45% $(r = M - 1)$ per decade.
If we set $x = 0$ for 2000, the year 2010 corresponds to $x = 1$ (one decade after 2000) and the year 2025 is $x = 2.5$ decades after 2000. Therefore our predictions are

$$y = 6.92 \cdot 0.7855 = 5.43$$

and

$$y = 6.92 \cdot 0.7855^{2.5} = 3.78$$

Both are given in deaths per 1,000 live births.

Ex.) The population of the City of Radford grew from 15,859 in 2000 to 16,408 in 2010. Assuming an exponential model, what is the annual growth rate?

$y = y_0 M^x$

$16408 = 15859 M^{(10)}$

$\frac{16408}{15859} = M^{10}$

$\left(\frac{16408}{15859}\right)^{\frac{1}{10}} = (M^{10})^{\frac{1}{10}}$

$1.0034 = M$

$r = M - 1 = 1.0034 - 1 = 0.0034$

The annual growth rate is approximately 0.34%.

Ex.) In 1890 Shakespeare enthusiasts released 60 European starlings in New York's Central Park. A century later its estimated North American population was 200 million (stanford.edu). What is the annual growth rate, and when will the population reach one billion?

$y = y_0 M^x$

$200000000 = 60 M^{(100)}$

$\frac{200000000}{60} = M^{100}$

$\left(\frac{200000000}{60}\right)^{\frac{1}{100}} = (M^{100})^{\frac{1}{100}}$

$1.1621 = M$

$r = M - 1 = 1.1621 - 1 = 0.1621$

The annual growth rate is approximately 16.21%.

To find when the population will reach one billion, start with the initial population of 200 million in the year 1990.

$y = y_0 M^x$

$1{,}000{,}000{,}000$
$= 200{,}000{,}000(1.1621)^x$

$\frac{1{,}000{,}000{,}000}{200{,}000{,}000} = (1.1621)^x$

$5 = (1.1621)^x$

$\log 5 = \log(1.1621)^x$

$\log 5 = x \log(1.1621)$

$\frac{\log 5}{\log(1.1621)} = x$

$10.713 = x$

In approximately another 11 years, the population will reach 1 billion. Therefore, in the year 2001 the starling population will exceed 1 billion.

Ex.) Elephantdatabase.org reports that the total population of elephants in Africa declined from 472,134 elephants in 2007 to 421,955 animals in 2012. Assuming an exponential decay, predict when the population will be down to 350,000 elephants.

$y = y_0 M^x$

$421{,}955 = 472{,}134M^{(5)}$

$\frac{421{,}955}{472{,}134} = M^5$

$\left(\frac{421{,}955}{472{,}134}\right)^{\frac{1}{5}} = (M^5)^{\frac{1}{5}}$

$0.9778 = M$

To find when the population will be down to 350,000, start with the initial population of 472,134 in 2007.

$y = y_0 M^x$

$350{,}000 = 472{,}134(0.9778)^x$

$\frac{350{,}000}{472{,}134} = (0.9778)^x$

$\log\frac{350{,}000}{472{,}134} = \log(0.9778)^x$

$\log\frac{350{,}000}{472{,}134} = x\log(0.9778)$

$\frac{\log\frac{350{,}000}{472{,}134}}{\log(0.9778)} = x$

$13.333 = x$

In 2020 the elephant population will be down to 350,000.

3. Doubling Time

Doubling time is the time it takes to double a value in an exponential model.

Given any initial population, say y_0, twice this value is $2y_0$.

To determine the doubling time, solve

$$2y_0 = y_0 M^x$$

for x.

$$2 = M^x$$
$$\log 2 = x \log M$$
$$x = \frac{\log 2}{\log M}$$

Recall, $M = 1 + r$.

Ex.) Suppose a population of rats is growing at an annual rate of 12%. Estimate how long it would take for the rat population to double.

$M = 1 + r = 1 + 0.12 = 1.12$

$x = \frac{\log 2}{\log M} = \frac{\log 2}{\log 1.12} = 6.116 \ldots$

After approximately 6.12 years the rat population doubled.

Ex.) A lung cancer cell doubles about every three months (mitosis). The cancer is detected in X-rays when about 1 billion cells are present. Beginning with a single malignant cell, how long does it take until the cancer is detected?

Let's measure time in years. In this case we know the doubling time,

$$x = 3\ months = 0.25\ years$$

What do not have is the growth rate or growth multiplier.

$x = \frac{\log 2}{\log M}$

$0.25 = \frac{\log 2}{\log M}$

$\log M = \frac{\log 2}{0.25}$

$10^{\log M} = 10^{\left(\frac{\log 2}{0.25}\right)}$

$M = 16$

The model for this problem is then

$y = 1(16)^x$

The cancer is detected when there are 1 billion cells, i.e., find x when $y = 1{,}000{,}000{,}000$

$1{,}000{,}000{,}000 = 1(16)^x$

$\log 1{,}000{,}000{,}000 = \log(16)^x$

$9 = x \log 16$

$\frac{9}{\log 16} = x$

$7.474 \ldots = x$

After approximately 7.5 years the cancer is detectable.

4. Half-Life

Half-Life is the time it takes for a value to be reduced by one-half. As we did with doubling time, we can develop a formula to determine the half-life of a quantity.

Given any initial value, say y_0, one-half this value is then $\frac{1}{2}y_0$.

To determine the half-life time, solve

$$\frac{1}{2}y_0 = y_0 M^T$$

for T.

$$\frac{1}{2} = M^T$$

$$\log\frac{1}{2} = T\log M$$

$$T = \frac{\log\frac{1}{2}}{\log M}$$

Ex.) Suppose a radioactive isotope of an element was placed in storage, decaying at a rate of 6.25% annually. How long would it take for only one-half of the element to remain?

Recall, $M = 1 + r = 1 - 0.0625 = 0.9375$

The half-life of the element is then,

$$\mathrm{T} = \frac{\log\frac{1}{2}}{\log 0.9375} = 10.7400\ \ldots$$

After approximately 10.74 years, half of the element will remain.

Ex.) The half-life of the radioactive isotope ^{14}C is 5730 years (abundance $^{14}C{:}^{12}C = 1{:}10^{12}$). A piece of wood measures only 20% of the common $^{14}C{:}^{12}C$ ratio. How old is it?

In this case the half-life is known.

$$T = \frac{\log \frac{1}{2}}{\log M}$$

$$5730 = \frac{\log \frac{1}{2}}{\log M}$$

$$\log M = \frac{\log \frac{1}{2}}{5730}$$

$$10^{\log M} = 10^{\left(\frac{\log \frac{1}{2}}{5730}\right)}$$
$$M \approx 0.99988$$

The model for this problem is then
$$y = y_0(0.99988)^T$$

If the final amount desired is 20% of the original amount, we will solve

$$0.2\mathrm{y}_0 = \mathrm{y}_0(0.99988)^{\mathrm{T}}$$
for T.

$$0.2 = (0.99988)^{\mathrm{T}}$$
$$\log 0.2 = \log(0.99988)^T$$
$$\log 0.2 = T\log(0.99988)$$
$$\frac{\log 0.2}{\log(0.99988)} = T$$
$$13{,}411.2 \approx T$$

The wood is approximately 13,411 years old.

5. Logarithmic Scale

Using logarithms, let's rewrite the basic exponential equation, $y = y_0 M^x$.

$$\log y = \log(y_0 M^x)$$
$$\log y = \log(y_0) + \log(M^x)$$
$$\log y = \log(M) \cdot x + \log(y_0)$$

Thus,
$$y = y_0 M^x \Leftrightarrow \log y = \log(M) \cdot x + \log(y_0)$$

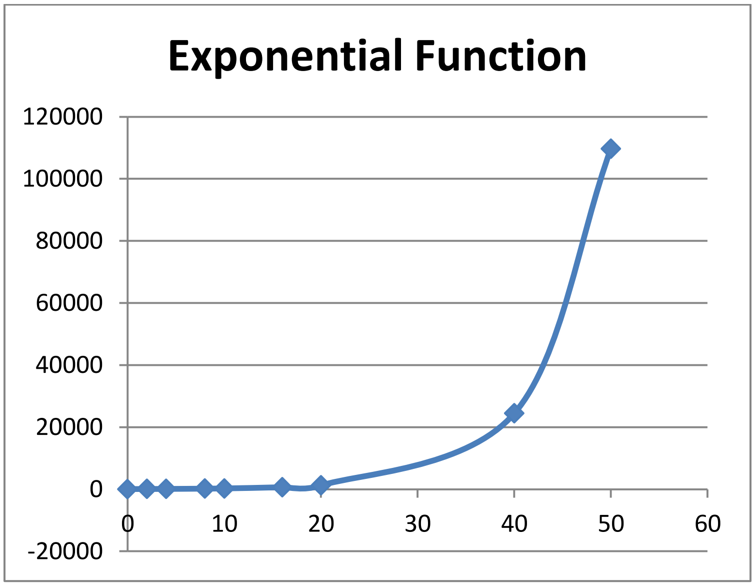
If y is an exponential function, then the logarithm of y is a linear function with slope $m = \log(M)$ and y-intercept $b = \log(y_0)$.

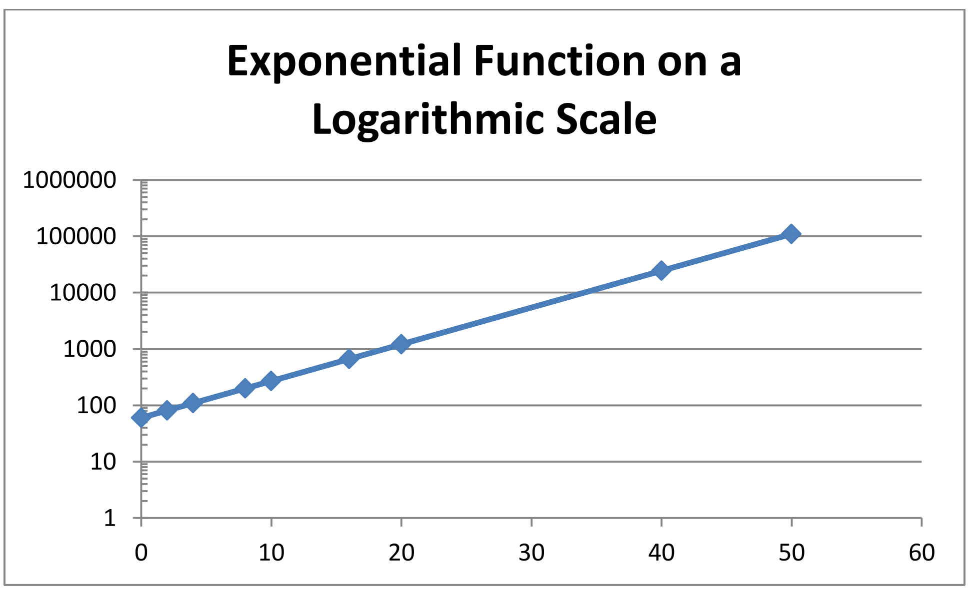
By changing the scales of the typical *x*-*y* plane to logarithmic scales, an exponential function looks like a straight line.

For instance, consider the exponential function $y = 60(1.1621)^x$ (from the starling example earlier).

Graphing, with Excel, one can see the difference

x	0	2	4	8	10	16	20	40	50
y	60	81.03	109.4	199.6	269.5	663.8	1210.7	24428.2	109730





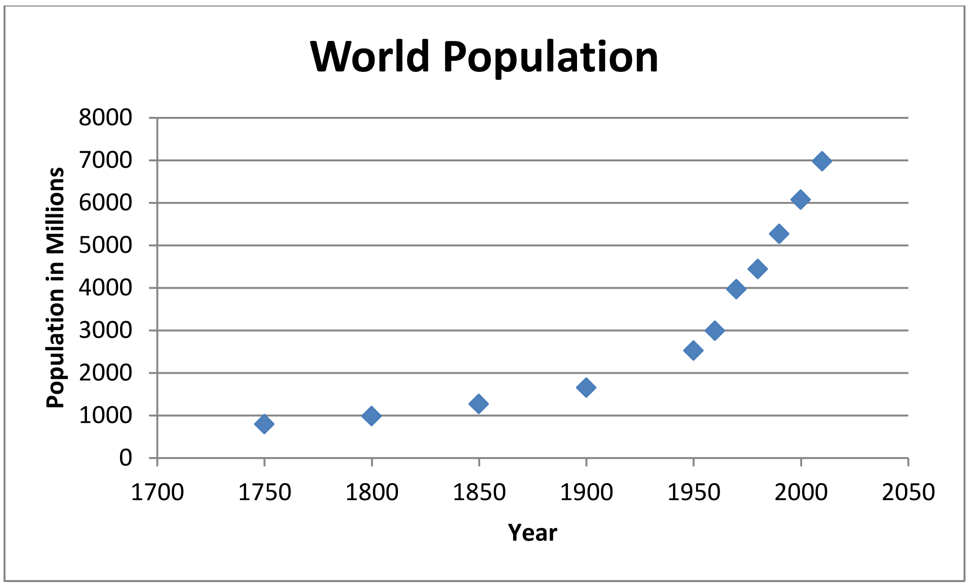
Ex.) The world population (in millions) is given in the table to the right. Plot the world population as a function of the year, Y, then graph using a logarithmic scale.

Year	Population (Millions)
1750	791
1800	978
1850	1262
1900	1650
1950	2519
1960	2982
1970	3962
1980	4435
1990	5263
2000	6070
2010	6972

You can use Excel to assist.
Begin by constructing the table. You may either use rows or columns (as displayed in the current table).

Highlight the entries you wish to display on the graph. Click on "Insert" and find the "Scatter" button. Select the style of your choice. Here, let's just plot the points ("Scatter with only Markers").

To label the axes, click on the graph. "Chart Tools" should then appear. Click on "Layout" and select "Axis Titles." Choose the axis you wish to label. Let's use *Year* for the horizontal axis and *Population in Millions* for the vertical axis.



Before plotting with the logarithmic scale, recall the population is given as a value in the millions (i.e., 791 on the table corresponds to 791 million = 791,000,000). Convert each value by multiplying by 1,000,000, then take the logarithm of each (base 10).

To plot using a logarithmic scale, graph as done previously using the values from the logarithm found.