Probability

1. Probability

Probabilities are used to express the chance of occurrence of some event (a hurricane making landfall, a king being drawn in poker, or choosing the correct door in *Let's Make a Deal*).

Determining probabilities of many natural phenomena is complex, but other scenarios are more simplistic. In this section we will focus on some easier examples to get used to probabilities.

Let E represent an event or successful outcome and let S represent the ***sample space*** or all possible outcomes.

Then the probability of the event E occurring is

$$P(E) = \frac{n(E)}{n(S)} = \frac{\#\ of\ successful\ outcomes}{\#\ of\ possible\ outcomes} = \frac{\#\ of\ specific\ outcomes}{total\ \#\ of\ possible\ outcomes}$$

Here, $n(S)$ must be finite and all outcomes are equally likely (they have the same chance of occurring).

Probabilities can be expressed as fractions, decimals, or percentages.

Let's begin with a simple example.

Ex.) Taking a standard quarter in hand, what is the probability of getting tails?

There are two possible outcomes, flipping a heads (H) or flipping a tails (T).

Thus, $S = \{H, T\}$ and $E = \{T\}$

$$P(E) = \frac{n(E)}{n(S)} = \frac{1}{2}$$

Therefore, the probability of flipping a tails is 1 in 2 or 50%.

Ex.) What is the probability of rolling a "3" on a fair die?

There are six possible outcomes, rolling a "1," "2," "3," "4," "5," or "6."

Thus, $S = \{1,2,3,4,5,6\}$ and $E = \{3\}$

$$P(E) = \frac{n(E)}{n(S)} = \frac{1}{6}$$

Therefore, the probability of rolling a "3" is 1 in 6 or approximately 16.67%

Ex.) What is the probability of drawing a black three from a standard deck of cards?

There are fifty-two possible outcomes, drawing any card from a standard deck of cards, but only two are black threes: the three of spades or the three of clubs.

$$P(E) = \frac{n(E)}{n(S)} = \frac{2}{52} = \frac{1}{26}$$

Therefore, the probability of drawing a black three is 1 in 26 or approximately 3.85%

Ex.) You roll a pair of dice. What is the probability of rolling an 8 or higher?

First roll/ second roll	**1**	**2**	**3**	**4**	**5**	**6**
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

There are a total of 36 possible outcomes, and 15 of the outcomes (highlighted in yellow) lead to a total of 8 or higher.

$$P(E) = \frac{n(E)}{n(S)} = \frac{15}{36} = \frac{5}{12}$$

Therefore, the probability of rolling an 8 or higher is 5 in 12 or approximately 41.67%

Ex.) You flip a coin three times. What is the probability of getting exactly two heads?

$$S = \left\{ \begin{matrix} HHH, HHT, HTH, HTT, \\ THH, THT, TTH, TTT \end{matrix} \right\}$$

$$P(E) = \frac{3}{8} = 0.375$$

Therefore, the probability of getting exactly two heads is 3 in 8 or 37.5%

2. Probability Rules

The following rules about probabilities are quite useful:

(i) The probability of any event is always between 0 and 1. (i.e., between 0% and 100%) You can use the compound inequality:

$$0 \leq P(event) \leq 1$$

(ii) The probability an event will *not* occur is equal to 1 minus the probability of the event occurring:

$$P(not\ event) = 1 - P(event)$$

From the last example, the probability of *not* getting two heads is

$$\begin{aligned} P(not\ 2\ heads) &= 1 - P(2\ heads) \\ &= 1 - 0.375 = 0.625 \end{aligned}$$

or, 67.5%.

3. Probability Types

(1) Experimental Probability or Observed Probability

This is the probability by actual measurements from an experiment. For instance, suppose you roll a pair of 6's 27 out of a 100 throws of two dice. The experimental probability is then

$$\frac{27}{100} = 0.27 = 27\%.$$

(2) *Theoretical Probability*

This is the "true" probability of an event happening.

Law of Large Numbers

As a procedure is repeated over and over, the experimental probability will approach the true probability.

This law is useful for situations where calculating probabilities of natural events (tornadoes, landslides, hurricanes, etc.) is next to impossible because we simply do not understand the processes involved well enough to make accurate predictions. Instead, the best that can be done is to estimate such probabilities using historical data and the law of large numbers.

4. Using Punnett Squares

Punnett squares can be used as a tool for problems of probability and genetics.

Ex.) Suppose that for one gene we have alleles A and a. You cross two heterozygotes (Aa x Aa), what is the probability that a progeny has genotype aa?

You can use a Punnett Square to determine the possible progenies, or the sample space.

	A	a
A	AA	Aa
a	Aa	aa

$$P(aa) = \frac{1}{4} = 0.25 = 25\%$$

Ex.) If a woman is homozygous normal and her husband is homozygous for a genetically inherited recessive disease and they decide to become parents, what is the probability that the child will have the recessive allele?

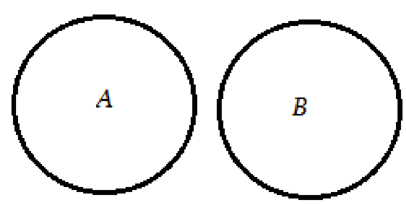
You can use a Punnett Square to determine the possible progenies.

	A	A
a	Aa	Aa
a	Aa	Aa

$$P(child\ has\ the\ recessive\ allele) = \frac{4}{4} = 1 = 100\%$$

5. Exclusive vs. Independent

For *mutually exclusive* events A and B (the occurrence of one event precludes the other), the probability of A **or** B is the sum of the individual probabilities:



$$P(A\ \textbf{or}\ B) = P(A) + P(B)$$

Because the events are ***mutually exclusive***, the probability of A **and** B is zero (they both cannot happen at the same time).

$$P(A\ \textbf{and}\ B) = 0$$

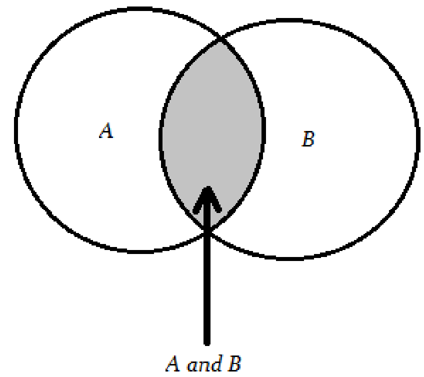
For *independent* events A and B (the occurrence of one event is independent of the other, or the outcome of one event has no influence on the other), the probability of the distinct events occurring successively or jointly is the product of their probabilities.

$$P(A \textbf{ and } B) = P(A) \cdot P(B)$$

However, since the events are independent,

$$P(A \textbf{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

The reason $P(A \text{ and } B)$ is subtracted is that $P(A \text{ and } B)$ is counted *twice*, once in $P(A)$ and again in $P(B)$.



Ex.) Suppose there are 18 students who study math, 24 students who study biology and there are 34 students all together.

(i) What is the probability a randomly chosen student studies math or biology?

The two events are not mutually exclusive (a student can study both math and biology).

Let n represent the number of students that study both math and biology. This implies

$$18 - n = \text{the number of students that only study math}$$

$$24 - n = \text{the number of students that only study math}$$

There are 34 students total, so

$(18 - n) + n + (24 - n) = 34$
$42 - n = 34$
$n = 8$

Thus, there are 10 students that only study math and 16 that only study biology.

$$P(math \textbf{ or } biology) = P(math) + P(biology) - P(math \text{ and } biology)$$
$$= \frac{18}{34} + \frac{24}{34} - \frac{8}{34} = \frac{34}{34} = 100\%$$

(ii) What is the probability a randomly chosen student studies math and biology?

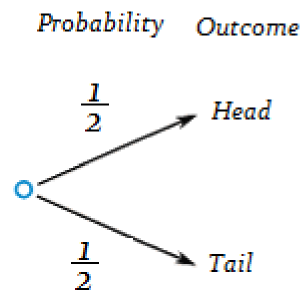
$$P(math \textbf{ and } biology) = \frac{\#of\ students\ that\ study\ both}{\#of\ students} = \frac{8}{34} = \frac{4}{17} \approx 23.53\%$$

6. Tree Diagrams

Calculating probabilities can be hard - sometimes you add them, sometimes you multiply them, and often it is hard to figure out what to do.

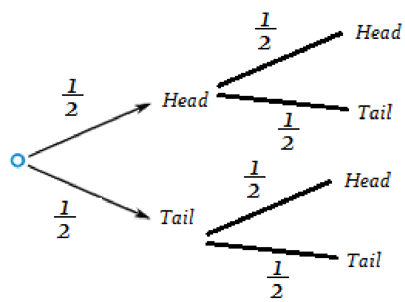
Ex.) You toss a coin twice. What is the probability of two heads?

One option is to construct a table (as we did in the last class session). Another option is to construct a tree diagram.



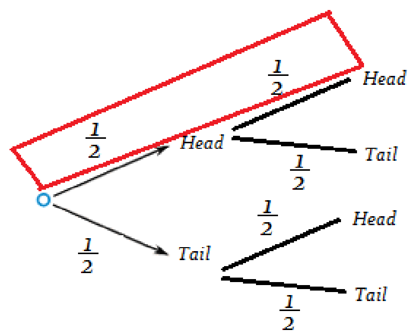
Here there are two branches: heads and tails
The probability of each occurring on the first toss is $\frac{1}{2}$.

Next extend the tree diagram to represent the second toss.



To calculate ches.

$P(2\ heads) = \frac{\ }{2}\cdot\frac{\ }{2} = \frac{\ }{4}$



Ex.) A family has three children. What is the probability that

(i) all three are boys?

$$P(3\ boys) = \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2} = \frac{1}{8}$$

(ii) the first is a boy, the second is a girl, and the third is a boy?

Note the order is specific here.

$$P(bgb) = \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2} = \frac{1}{8}$$

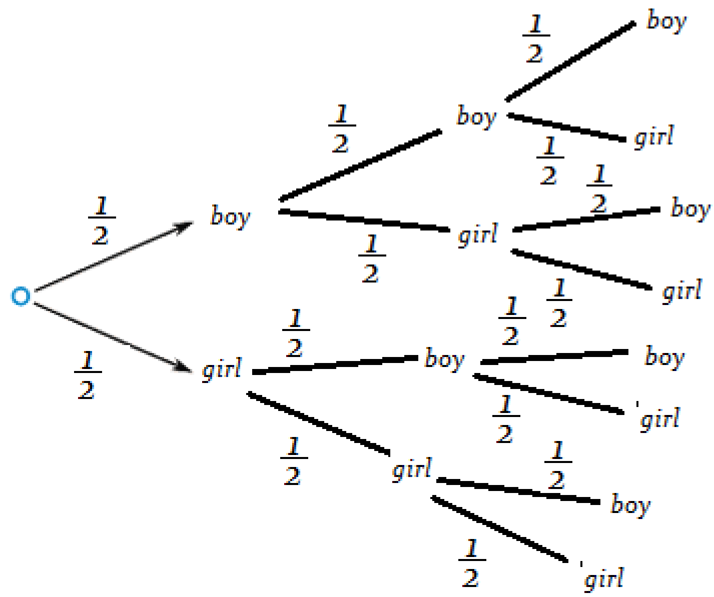
Although order is specific, each outcome is independent of the previous. Thus, *in this case*, the likelihood of any order of children (bbg, gbg, ggg, etc.) is the same.

(iii) at least one child is a girl?

$$P(at\ least\ one\ girl) = \frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8} = \frac{7}{8}$$

or

$$P(at\ least\ one\ girl) = 1 - P(3\ boys) = 1 - \frac{1}{8} = \frac{7}{8}$$



Although the previous examples contained independent events, tree diagrams can be used to calculate probabilities of mutually exclusive events.

Ex.) A disease is linked to a recessive allele. Your aunt (mother's sister) has the disease, but neither do you, your mother nor your maternal grandparents. For simplicity assume that no one on your dad's side of the family carries the disease (and your father doesn't carry the allele). What is the probability that you carry the allele (and might pass it on to your children)?

Because your mother's sister has the disease, the aa progeny must exist.

Maternal Grandparents		
		a
a		aa

However, because your mother's parents do not have the disease, they must be either AA or Aa (they cannot be aa, otherwise they have the disease). Couple this fact with the previous and the only possibility is the maternal grandparents are heterozygotes.

Maternal Grandparents		
	A	a
A	AA	Aa
a	Aa	aa

So, which is your mother? There are 4 possible outcomes, but only three are disease free. Therefore, for your mother,

$$P(AA) = \frac{1}{3} \text{ and } P(Aa) = \frac{2}{3}.$$

For your father (he does not have the allele)

$$P(AA) = 1$$

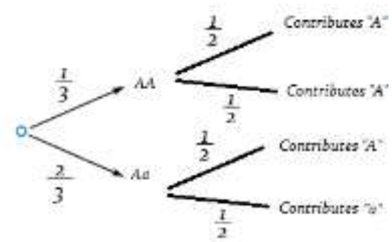
If both parents are homozygotes (AA), then their progeny will also be homozygotes:

	A	A
A	AA	AA
A	AA	AA

If your mother carries the allele (Aa), then the progeny include

	A	a
A	AA	Aa
A	AA	Aa

From a tree diagram, you can determine what the probability is that you carry the allele:



Thus the probability you carry the allele is the probability your mother carries the allele and passes it on to her progeny:

$$\text{P(Mother is Aa and progeny Aa)} = \frac{2}{3} \cdot \frac{1}{2} = \frac{1}{3}$$

7. Replacement vs. Without Replacement

If you replace the first selection from a population before making a second selection, then the probability of making any given selection is unchanged. On the other hand, if the first selection is not replaced, then the probability of making the second selection changes.

Ex.) In group of mice there are 75 white mice, 50 brown mice, and 25 black mice.

(i) What is the probability of selecting a brown mouse?

There are a total of $75 + 50 + 25 = 150$ mice. The probability of selecting a brown mouse is

$$P(brown) = \frac{50}{150} = \frac{1}{3} \approx 0.33333 = 33.33\%$$

(ii) You select two mice, one at a time with replacement. What is the probability they are both brown?

$$P(2\ brown) = \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9} \approx 0.1111 = 11.11\%$$

(iii) You select three mice, one at a time without replacement. What is the probability of selecting three black mice?

$$P(3\text{ black}) = \frac{25}{150} \cdot \frac{24}{149} \cdot \frac{23}{148} = \frac{13800}{3307800} = \frac{23}{5513} = 0.00417195719\ldots \approx 0.42\%$$

(iv) You draw a pair, one at a time, without replacement. What is the probability of not drawing a white mouse?

Pair choices:

	W	Br	Bk
W	WW	WBr	WBk
Br	BrW	BrBr	BrBk
Bk	BkW	BkBr	BkBk

$$P(Pair, neither\ white) = P(BrBr) + P(BrBk) + P(BkBr) + P(BkBk)$$

$$= \frac{50}{150} \cdot \frac{49}{149} + \frac{50}{150} \cdot \frac{25}{149} + \frac{25}{150} \cdot \frac{50}{149} + \frac{25}{150} \cdot \frac{24}{149}$$

$$= \frac{2450}{22350} + \frac{1250}{22350} + \frac{1250}{22350} + \frac{600}{22350}$$

$$= \frac{2450 + 1250 + 1250 + 600}{22350}$$

$$= \frac{5550}{22350} = \frac{111}{447} = \frac{37}{149} \approx 0.2483221476510067 \approx 24.83\%$$

Alternatively, you could calculate the result as
$P(Pair, neither\ white) = 1 - P(at\ least\ one\ W)$

$$= 1 - \left(P(WW) + P(WBr) + P(WBk) + P(BrW) + P(BkW)\right)$$

$$= 1 - \left(\frac{75}{150} \cdot \frac{74}{149} + \frac{75}{150} \cdot \frac{50}{149} + \frac{75}{150} \cdot \frac{25}{149} + \frac{50}{150} \cdot \frac{75}{149} + \frac{25}{150} \cdot \frac{75}{149}\right)$$

$$= 1 - \left(\frac{5550}{22350} + \frac{3750}{22350} + \frac{1875}{22350} + \frac{3750}{22350} + \frac{1875}{22350}\right)$$

$$= 1 - \left(\frac{16800}{22350}\right)$$

$$= \frac{22350}{22350} - \left(\frac{16800}{22350}\right) = \frac{111}{447} = \frac{37}{149} \approx 0.2483221476510067 \approx 24.83\%$$

$$= \frac{5550}{22350}$$

Ex.) The table below details the magnitude 7 (or higher) earthquakes that occurred along the San Andreas fault during the 20th century. What is the probability that a magnitude 7 or greater earthquake will take place any given year in the twenty-first century along the San Andreas fault?

Year	Mag	Year	Mag	Year	Mag	Year	Mag	Year	Mag
1906	8.3	1923	7.2	1940	7.1	1980	7.2	1992	7.3
1922	7.3	1927	7.3	1952	7.7	1989	7.1	1999	7.1

During the 20th century there were 10 earthquakes of magnitude or greater. In that century there were 100 possible years such an earthquake could occur.

$$P(quake\ occurs\ in\ the\ 20th\ century) = \frac{actual\ years}{possible\ years} = \frac{10}{100} = 0.1 = 10\%$$

There was a 10% chance a large earthquake occurred in any given year in the 20th century. Thus, one could make an argument that the same probability holds true for the 21st century.

7. Kin Altruism & Hamilton's Rule[1]

Kin selection is the evolutionary strategy that favors the reproductive success of an organism's relatives, even at a cost to the organism's own survival and reproduction. Kin altruism is altruistic behavior whose evolution is driven by kin selection. Kin selection is an instance of ***inclusive fitness***, which combines the number of offspring produced with the number of genetically related individuals that can be indirectly produced by supporting others, such as siblings.

In humans, altruism is more likely and stronger with kin than with unrelated individuals; for example, humans give presents according to how closely related they are to the recipient. In other species, vervet monkeys use allomothering, where related females such as older sisters or grandmothers often care for young, according to their relatedness. The social shrimp *Synalpheus regalis* protects juveniles within highly related colonies.

In 1964 W. D. Hamilton addressed the question of how altruistic behavior may have evolved, given a conflict of ***individual fitness*** (the relative ability of an individual took into account to survive, reproduce and propagate genes in an environment.), in quantitative terms. Hamilton took the degree of relatedness when addressing altruistic social encounters.

Let

- $B =$ benefit, average extra offspring (who is in danger)
- $C =$ cost, fewer offspring by altruist (who is coming to the rescue)
- $r =$ relatedness coefficient, number of shared genes on average (0.5 for siblings, 0.25 for aunt/niece, 0.125 for cousins)

Altruism is favored if $rB > C$.

Imagine a situation in which the threat of predation can evoke one of two possible responses in a vigilant monkey: give an alarm call, whereupon the caller will incur a cost (greater chance of being eaten) at the benefit of the others; or, remain silent, whereupon some of the other members will be eaten while the actor escapes unharmed.

If the beneficiaries of this act of ultimate altruism are close relatives, it would actually benefit the *inclusive fitness* of that individual to give the call and save their kin rather than themselves. Because relatives will have a higher tendency to share similar genetic material, the total genetic benefit for this behavioral strategy would exceed a selfish strategy. Therefore, natural selection has provided a mode by which group fitness is augmented by individuals acting, really, in their own self-interest.

Ex.) A monkey is under attack by a predator, it has a 75% chance of survival if the predator is distracted. The monkey's brother has a 25% chance of dying if he distracts the predator. Both would have an average number of 4 offspring. Should the brother help?

For this problem,
$B = 4 \cdot 75\% = 4 \cdot 0.75 = 3$
$C = 4 \cdot 25\% = 4 \cdot 0.25 = 1$
$r = 0.5$ (sibling)
$rB = 0.5 \cdot 3 = 1.5 > 1 = C$
Because $rB > C$, by Hamilton's Rule, the brother should help.

Ex.) The same question as above, but replace the monkey under attack with two monkeys under attack.

$B = 2 \cdot (4 \cdot 75\%) = 2 \cdot (4 \cdot 0.75) = 6$
$C = 4 \cdot 25\% = 4 \cdot 0.25 = 1$
$r = 0.5$ (sibling)
$rB = 0.5 \cdot 6 = 3 > 1 = C$
Because $rB > C$, by Hamilton's Rule, the brother should help.

Ex.) A young man is close to drowning in heavy surf. Should his sister swim out and try to rescue him if she has a 25% risk of drowning herself? Assume that both would have two kids later in life.

For this problem,

$B = 2 \cdot 1 = 2$
$C = 2 \cdot 25\% = 2 \cdot 0.25 = 0.5$
$r = 0.5$ (sibling)
$rB = 0.5 \cdot 2 = 1 > 0.5 = C$
Because $rB > C$, by Hamilton's Rule, the sister should help.

Ex.) The same question as above, but replace the sister with a cousin.

In this case, B and C remain the same, but r becomes $r = 0.125$

$B = 2 \cdot 1 = 2$
$C = 2 \cdot 25\% = 2 \cdot 0.25 = 0.5$
$r = 0.125$ (cousin)
$rB = 0.125 \cdot 2 = 0.25 < 0.5 = C$
Because $rB < C$, by Hamilton's Rule, the cousin should not help.

Sources:
(1) http://www.wwnorton.com/college/anthro/bioanth/ch8/chap8.htm

Additional background: http://www.youtube.com/watch?v=unUG0BPMaIw