Proportions

1. Proportions

By definition, a ***proportion*** is the equality of two ratios.

$$\frac{a}{b} = \frac{c}{d}$$

Thus, two quantities y and x are proportional if their ratios remain constant. To symbolize this we use the notation

$$y \propto x$$

When speaking of ***direct proportions***, they change in the same fashion. For instance, if you double one quantity (x), then the other (y) will double as well. For that reason we say that "y ***varies directly*** with x" or "y is ***directly proportional*** to x."

If $y \propto x$, it follows that $\frac{y}{x} = k$ remains constant. Multiplying each side by x yields

$$y = kx.$$

The value of k, where $k = \frac{y}{x}$, is called the ***constant of proportionality***.

Some examples of direct proportionality include

- Linear motion: distance $\propto$ time
  distance $\propto$ speed
- Mass: amount in g $\propto$ amount in oz
- Medical Field:
  The flux in a blood vessel is proportional to the area of the cross section and to the velocity
  flux $\propto$ area $\cdot$ velocity

Some examples of items that are **not** in direct proportion include

- Temperature: Temperature in ℃ & Temperature in ℉
- Area: Area of a circle & its diameter
- Linear motion: time & speed

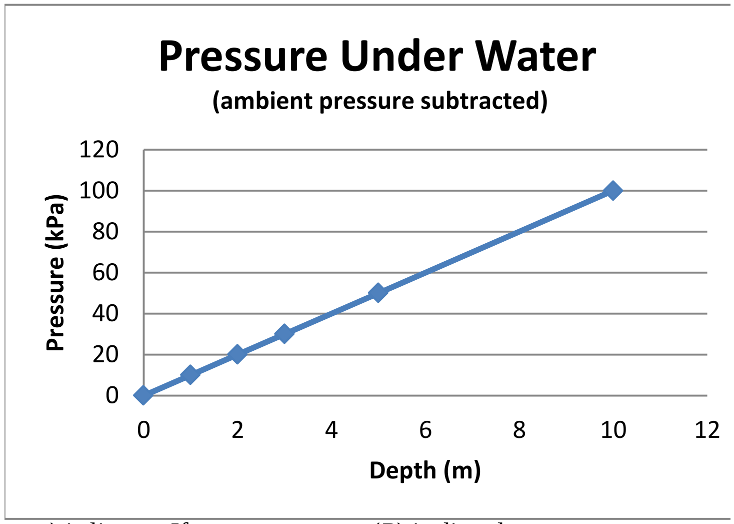
2. Direct Variation

If you subtract off the ambient air pressure, the pressure under water increases by about 10 kPa per 1 m of depth.

Thus for each 1 m of depth you dive down into the water, the water pressure increases by 10 kPa. As a table,

Depth (m)	Pressure (kPa)
0	0
1	10
2	20
3	30
5	50
10	100

Using a graphing program, Excel for instance, we can see a relationship that exists for this scenario:



From the graph we see the relationship between the two variables (depth and pressure) is linear. If we say pressure (P) is directly proportional to depth (d), then what equation can be written?

Because $P \propto d$, you can use the equation

$$P = kd$$

where k is the constant of proportionality.

With a line rising to the right, we expect the value of k to be positive (i.e., the line has positive slope). In this case, the value of k is 10 kPa/m, or

$$P = 10d.$$

This can be verified algebraically by selecting a point from the table (any point will do). When the depth was 5 m, the pressure was 50 kPa. Using $P = kd$ and substituting $P = 50$ and $d = 5$ allows you to find the value of k.

$$50 \text{ kPa} = k(5 \text{ m})$$

$$\frac{50 \text{ kPa}}{5 \text{ m}} = k$$

$$10 \frac{\text{kPa}}{\text{m}} = k$$

Thus, an important feature of direct variation or direct proportionality is the fact the relationship between the two variables are linear. When $k > 0$ you should expect growth (an incline) and when $k < 0$ you should expect decay (a decline).

Ex.) Suppose the weight, w, of an iguana is directly proportional to its length, l. If a 4-foot iguana weighs 20 pounds, then would is the expected weight of a 6-foot iguana?

$w \propto l \Rightarrow w = kl$
$20 \text{ lbs} = k(4 \text{ ft})$

$$\frac{20 \text{ lbs}}{4 \text{ ft}} = k$$

$5 \text{ lbs/ft} = k$

Thus the equation that is used in this problem is
$$w = 5l.$$

To answer the question, substitute $l = 6 \text{ ft}$

$$w = 5l = \frac{5 \text{ lbs}}{\text{ft}}(6 \text{ ft}) = 30 \text{ lbs}$$

The 6-ft iguana would be expected to weigh 30 lbs.

In the previous problem, the ratio between the iguana's weight and length is constant, or

$$k = \frac{w}{l}$$

regardless of the scenario. Thus,
$$k = \frac{w_1}{l_1} \text{ and } k = \frac{w_2}{l_2}.$$

Since both ratios are equivalent to k, they must in turn be equal to each other.

$$\frac{w_1}{l_1} = \frac{w_2}{l_2}$$

3. Solving Proportions

Once you have a proportion, you can apply the ***<u>Extremes-Means Property</u>***.

Extremes-Means Property

For a proportion of the form $\frac{a}{b} = \frac{c}{d}$, the product of the extremes is equal to the product of the means, or
$$ad = bc.$$

In other words, if you have a proportion, you can use ***<u>cross-multiplication</u>*** to solve.

Ex.) ([1]) A biological survey of the Ironwood Forest National Monument near Tucson was conducted between 2001 and 2003. One of the study plots in the Roskruge Mountains contained three large plants: the saguaro cactus, the ironwood tree, and the foothill palo verdes tree. In a second location within the forest the number of ironwood trees was counted. The table below indicates the density (number of trees per hectare).

Densities in Trees per Hectare	**Saguaro Cactus**	**Ironwood Tree**	**Foothill Palo Verdes**
Roskruge Mountains	315	270	165
Second Location	?	120	?

If the same proportions of large plants are found in both plots, what are the densities of saguaro cacti and foothill palo verdes trees?

$$\frac{\#\,Ironwood\ Trees\ in\ Plot\ 1}{\#\,Ironwood\ Trees\ in\ Plot\ 2} = \frac{\#\,Saguaro\ Cacti\ in\ Plot\ 1}{\#\,Saguaro\ Cacti\ in\ Plot\ 2}$$

$$\frac{270}{120} = \frac{315}{x} \Rightarrow 270x = (120)(315) \Rightarrow 270x = 37800 \Rightarrow x = \frac{37800}{270} = 140$$

$$\frac{\#\,Ironwood\ Trees\ in\ Plot\ 1}{\#\,Ironwood\ Trees\ in\ Plot\ 2} = \frac{\#\,Foothill\ Palo\ Verdes\ Trees\ in\ Plot\ 1}{\#\,Foothill\ Palo\ Verdes\ Trees\ in\ Plot\ 2}$$

$$\frac{270}{120} = \frac{165}{x} \Rightarrow 270x = (120)(165) \Rightarrow 270x = 19800 \Rightarrow x = \frac{19800}{270} \approx 73.33$$

There are 140 saguaro cacti and approximately 73 foothill palo verdes trees in the second plot.

4. Capture-Recapture

When trying to determine the size of a population, one could take a headcount, but not many species would be willing to hold still while you walk through counting each individual member of the species. Instead, biologists make use of rules involving proportions to estimate a population using the method of ***capture-recapture*** (also called ***mark-recapture***).

The general idea is to capture T animals, mark and release them. At a later date you capture n animals and find t of them have been marked. If the sample is representative of the population, then the ratio of marked animals in the population to marked in the sample should be the same as the ratio of the population size, N, to the size of the sample, or

$$\frac{T}{t} = \frac{N}{n}$$

Alternatively, the equation

$$\frac{T}{N} = \frac{t}{n}$$

may also be used (check the cross multiplication).

In the alternate equation, the left side describes what percent of the animals in the population were marked. Suppose you did a great, thorough job and marked 50% of the animals in your first capture session. Then, when you went back out, you would expect 50% of the second sample to have marks on them. In practice, we don't know N, the real population size, but we calculate N assuming the ratio of t to n is proportional to the ratio of T to N.

Using either equation and solving for N results in

$$Nt = nT \Rightarrow N = \frac{nT}{t}$$

Ex.) While on a biological survey on an island you capture, tag, and release 71 birds. Three days later you capture 83 birds, 23 of which are tagged. Estimate the total population.

Here, you are trying to determine the total population, N. In this problem, $T = 71$, $n = 83$, and $t = 23$.

$$N = \frac{nT}{t} = \frac{(83)(71)}{23} \approx 256.22$$

The total population of birds is approximately 256 birds.

Ex.) *Cephalorhynchus* hectori, Hector's dolphin, is found only in New Zealand. This dolphin is said to be the world's smallest (and possibly rarest). Because this dolphin is an inshore species with a limited home range, the majority of the population is found around Banks Peninsula – New Zealand's first marine mammal sanctuary (established in 1988).

Suppose at some point, 180 dolphins are marked originally (using photographs and the distinctive dorsal fins of each dolphin). A short time later, 7 dolphins are identifiable as ones "captured" earlier in a pod of 44 dolphins. Estimate the total dolphin population.

Here, you are trying to determine the total population, N. In this problem, $T = 180$, $n = 44$, and $t = 7$.

$$N = \frac{nT}{t} = \frac{(44)(180)}{7} \approx 1131.43$$

The total population is approximately 1131 dolphins.

(1) Langkamp, G and Hull, J. <u>*Quantitative Reasoning and the Environment: Mathematical Modeling in Context*</u>. 2007.