Section 4.1: Related Rates

Practice HW from Stewart Textbook (not to hand in)
p. 267 # 1-19 odd, 23, 25, 29

In a related rates problem, we want to compute the rate of change of one quantity in terms of the rate of change of another quantity (which hopefully can be more easily measured) at a specific instance in time.

Recall from Math 151! The instantaneous rate of change (from now on, we just shorten and call the rate of change) of a quantity is just its derivative with respect to <u>time</u> *t*. In our problems, we will differentiate quantities implicitly on both sides of an equation with respect to the variable *t*.

Example 1: Given $y = x^2 - 3x$, find $\frac{dx}{dt}$ given that $x = 1$ and $\frac{dy}{dt} = 2$.

Solution:

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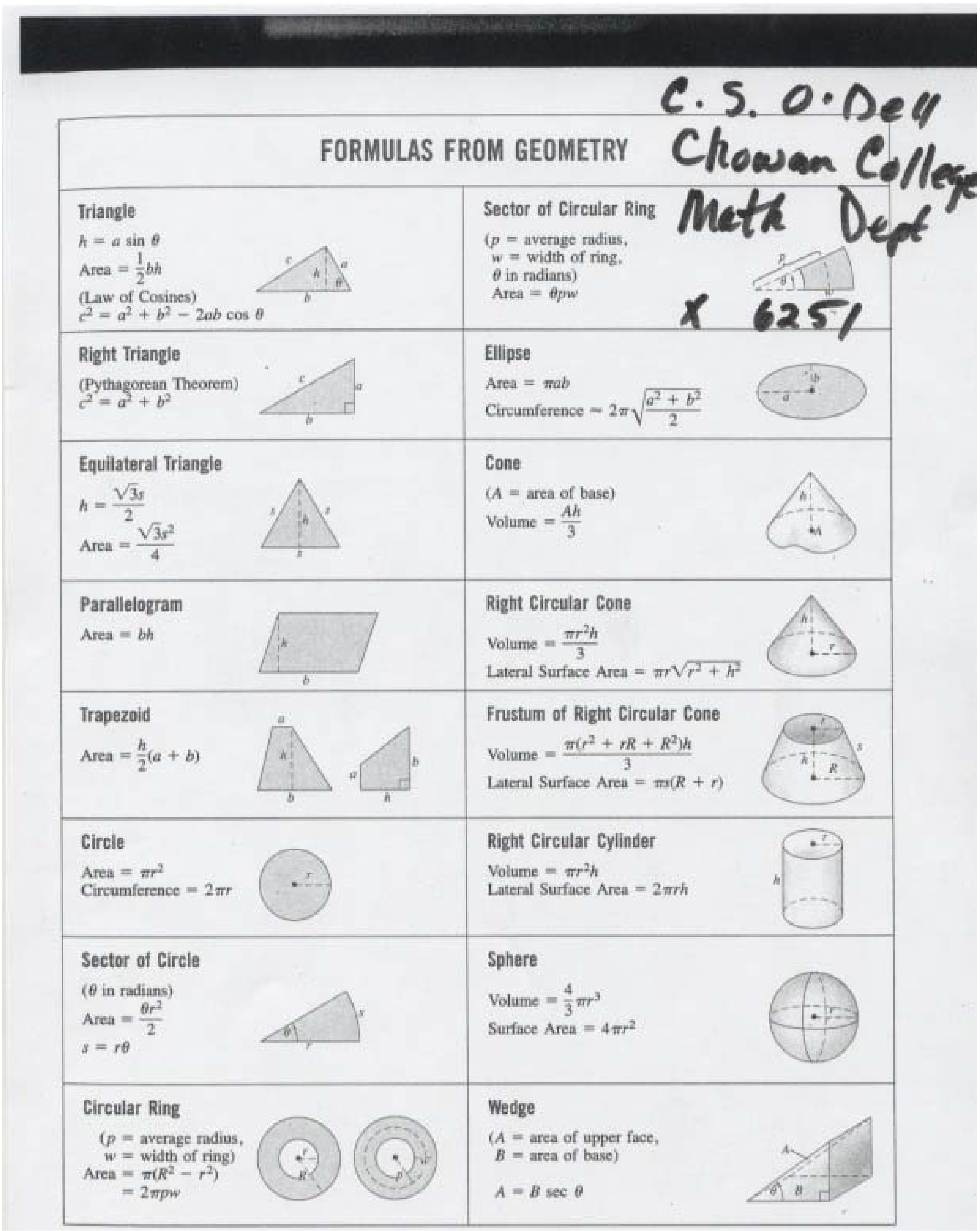
Steps for Setting up and Solving Related Rate Problems

1. After carefully reading the problem, ask yourself immediately what the problem wants you to find. Then determine the quantities that are given. Assign variables to quantities.
2. You will need to find an equation to obtain the quantities you obtained in step 1. This equation will be the one you will need to implicitly differentiate to obtain the variables rate of change quantities you determined in step 1. Drawing a picture sometimes will help. It may be necessary in some cases to use the geometry of the situation to eliminate one of the variables by substitution.
3. Implicitly differentiate both sides of the equation you found in step 2 with respect to time *t*. Solve for the unknown related rate, using the known quantities at specific instances in time for the quantities you assigned in step 1.

The following geometry formulas can sometimes be helpful.

Volume of a Cube: $V = x^3$, where x is the length of each side of the cube.
Surface Area of a Cube: $A = 6x^2$, where x is the length of each side of the cube.



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FORMULAS FROM GEOMETRY

Triangle
$h = a \sin \theta$
Area $= \frac{1}{2}bh$
(Law of Cosines)
$c^2 = a^2 + b^2 - 2ab \cos \theta$

Right Triangle
(Pythagorean Theorem)
$c^2 = a^2 + b^2$

Equilateral Triangle
$h = \frac{\sqrt{3}s}{2}$
Area $= \frac{\sqrt{3}s^2}{4}$

Parallelogram
Area $= bh$

Trapezoid
Area $= \frac{h}{2}(a + b)$

Circle
Area $= \pi r^2$
Circumference $= 2\pi r$

Sector of Circle
(θ in radians)
Area $= \frac{\theta r^2}{2}$
$s = r\theta$

Circular Ring
(p = average radius,
w = width of ring)
Area $= \pi(R^2 - r^2)$
$= 2\pi pw$

Sector of Circular Ring
(p = average radius,
w = width of ring,
θ in radians)
Area $= \theta pw$

Ellipse
Area $= \pi ab$
Circumference $\approx 2\pi\sqrt{\frac{a^2 + b^2}{2}}$

Cone
(A = area of base)
Volume $= \frac{Ah}{3}$

Right Circular Cone
Volume $= \frac{\pi r^2 h}{3}$
Lateral Surface Area $= \pi r\sqrt{r^2 + h^2}$

Frustum of Right Circular Cone
Volume $= \frac{\pi(r^2 + rR + R^2)h}{3}$
Lateral Surface Area $= \pi s(R + r)$

Right Circular Cylinder
Volume $= \pi r^2 h$
Lateral Surface Area $= 2\pi rh$

Sphere
Volume $= \frac{4}{3}\pi r^3$
Surface Area $= 4\pi r^2$

Wedge
(A = area of upper face,
B = area of base)
$A = B \sec \theta$

Example 2: A rock is dropped into a calm lake, causing ripples in the form of concentric circles. The radius of the outer circle is increasing at a rate of 2 ft/sec. When the radius is 5 ft, at what rate is the total area of the disturbed water changing.

Solution:

∎

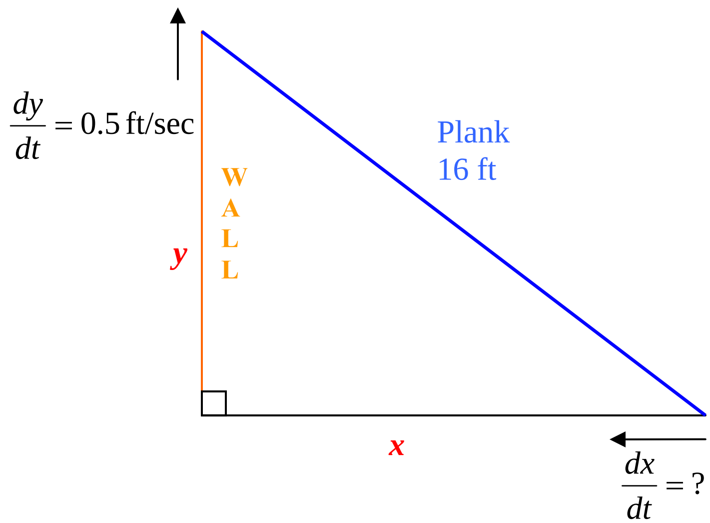
Example 3: Suppose the volume of spherical balloon is increasing at a rate of $288\ \text{in}^3/\text{min}$. Find the rate in which the diameter increases when the diameter of the balloon is 48 in.

Solution:

∎

Example 4: A construction worker pulls a 16 ft plank up the side of a building under construction by means of a rope tied to the end of a plank at a rate of 0.5 ft/s. How fast is the end of the plank sliding along the ground when it is 8 feet from the wall of the building?

Solution: In this problem, we want to find how fast (the rate) the ladder is sliding along the ground. The rate of change is given by the derivative with respect to time. We need to assign our variables. Consider the following figure.



Here, x = the distance the plank is from the wall, y = the distance the plank is up the wall from the ground. The plank is being pulled up the wall at a rate of 0.5 ft/s. This represents how much the height y is changing with respect to t, that is, $\frac{dy}{dt} = 0.5$. Our goal is the find how fast the plank is sliding along the ground, that is, the rate of change of x with respect to t, $\frac{dx}{dt}$. To relate the variable quantities, we use the Pythagorean Theorem. We see that

$$x^2 + y^2 = 16^2$$

or

$$x^2 + y^2 = 256$$

Differentiating both sides implicitly with respect to t gives

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

The next three steps solve this equation for $\frac{dx}{dt}$.

$$2x\frac{dx}{dt} = -2y\frac{dy}{dt}$$

$$\frac{dx}{dt} = -\frac{2y}{2x}\frac{dy}{dt}$$

$$\frac{dx}{dt} = -\frac{y}{x}\frac{dy}{dt}$$

We want to find $\frac{dx}{dt}$ at the precise point in time when $x = 8$ feet. However, we still do not know what y is in this precise point in time. We can find this using the Pythagorean Theorem. That is, if

$$x^2 + y^2 = 256$$
$$(8)^2 + y^2 = 256$$
$$y^2 = 256 - 64$$
$$y^2 = 196$$
$$y = \sqrt{196} = 14$$

Thus, we want $\frac{dx}{dt}$ at the precise point in time when $x = 8$ ft, $y = 14$ ft, and $\frac{dy}{dt} = 0.5\,\text{ft/s}$.
Hence, we have

$$\boxed{\frac{dx}{dt} = -\frac{y}{x}\frac{dy}{dt} = -\frac{14\,\text{ft}}{8\,\text{ft}}(0.5\,\text{ft/sec}) = -\frac{7}{8}\,\text{ft/sec} = \text{-}0.875\,\text{ft/s}}$$

Hence, the plank is moving towards the wall at a rate of -0.875 ft/s (the negative sign just means the distance between the plank and the wall is decreasing).

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Example 5: This problem is Exercise 15 on p. 267. A man starts walking north at 4 ft/s from a point *P*. Five minutes later a woman starts walking south at 5 ft/s from a point 500 ft due east of *P*. At what rate are the people moving apart 15 min after the woman starts walking?

Solution:

[redacted]

Example 6: A water tank has the shape of an inverted circular cone with base radius 6 ft and height 12 ft. If water is being pumped into the tank at a rate of $3\,\text{ft}^3/\text{s}$, find the rate the water level is rising when the water is 2 ft deep.

Solution:

[redacted]