Section 4.2: Maximum and Minimum Values

Practice HW from Stewart Textbook (not to hand in)
p. 276 # 1-7 odd, 15-19 odd, 23-29 odd, 33-45 odd

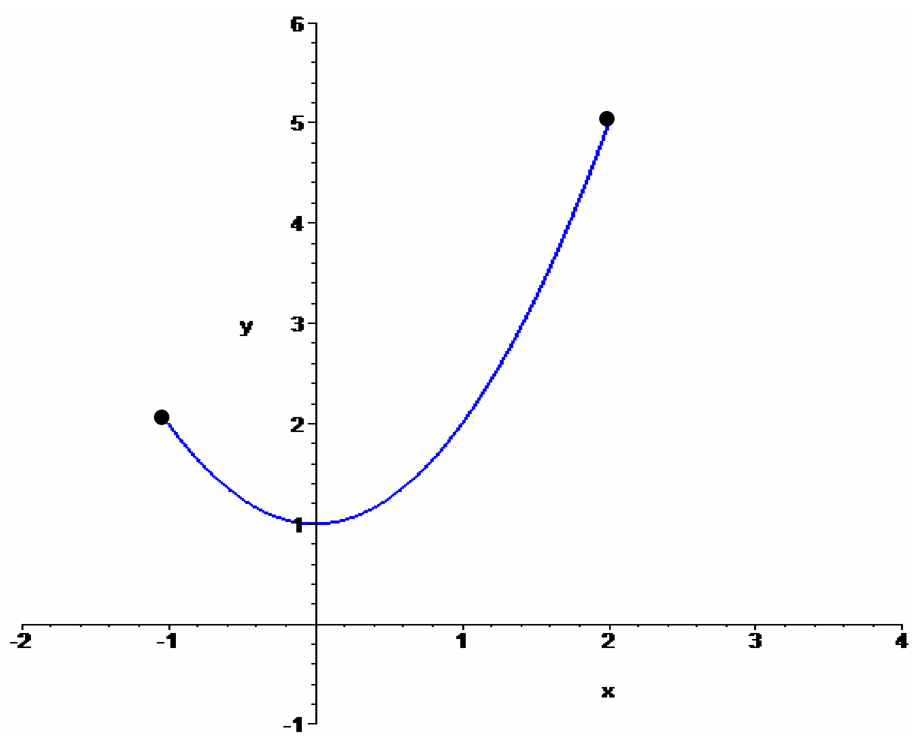
Extrema

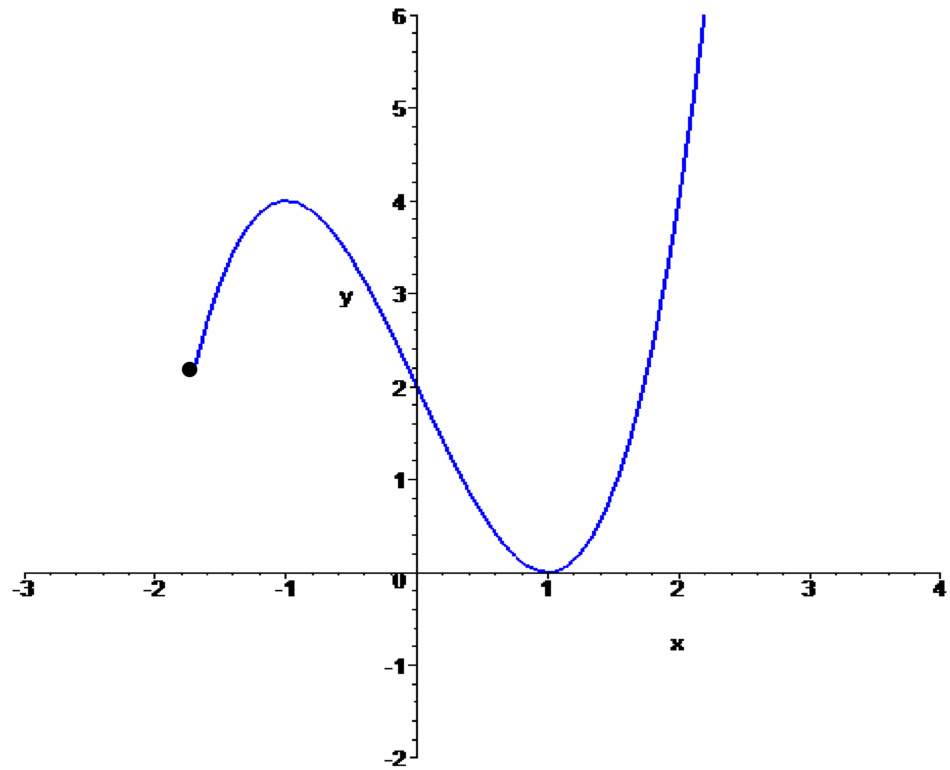
Let D be the domain of a function f.

1. A function f has and absolute maximum (global maximum) at $x = c$ if $f(c) \geq f(x)$ for all x in D ($f(c)$ is the largest y value for the graph of f on the domain D).
2. A function f has and absolute minimum (global minimum) at $x = c$ if $f(c) \leq f(x)$ for all x in D ($f(c)$ is the smallest y value for the graph of f on the domain D).

The absolute maximum and absolute minimum values are known as extreme values.

Example 1: Determine the absolute maximum and minimum values for the following graphs.





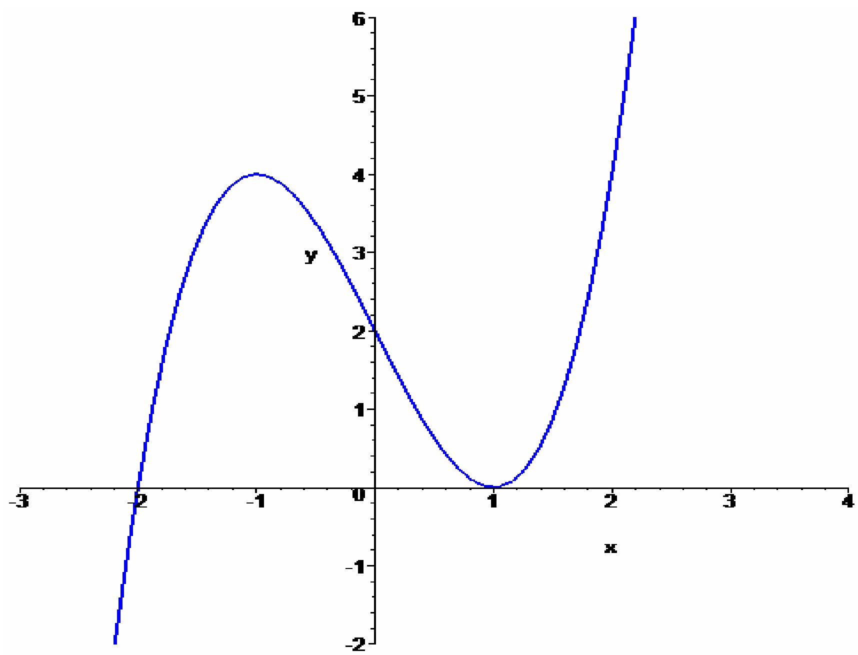
Relative Extrema

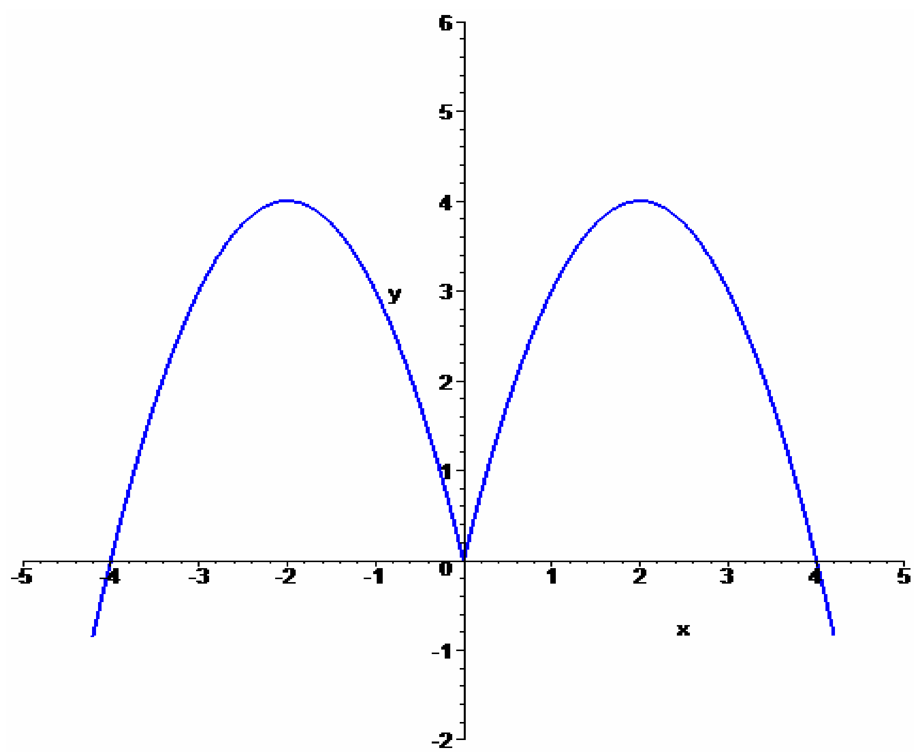
A function f has a local maximum (relative maximum) at $x = c$ if $f(c) \geq f(x)$ when x is near c (f changes from increasing to decreasing) at the point $(c, f(c))$.

A function f has a local minimum (relative minimum) at $x = c$ if $f(c) \leq f(x)$ when x is near c (f changes from decreasing to increasing) at the point $(c, f(c))$.

Example 2: Determine the local (relative) maximum and minimum points for the following graphs.

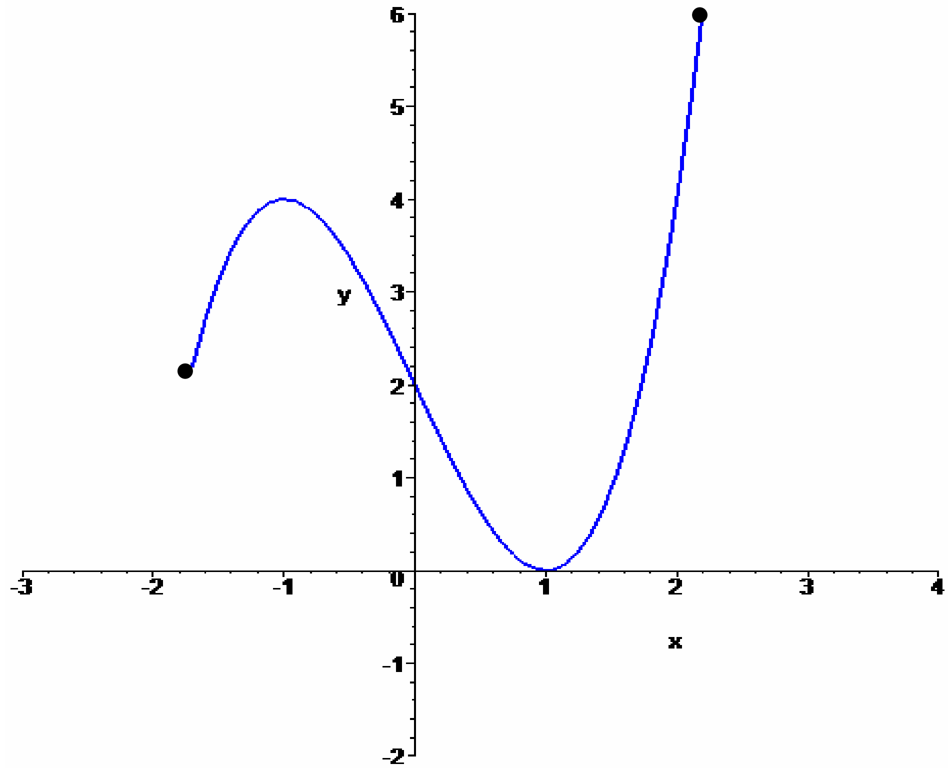
Solution:





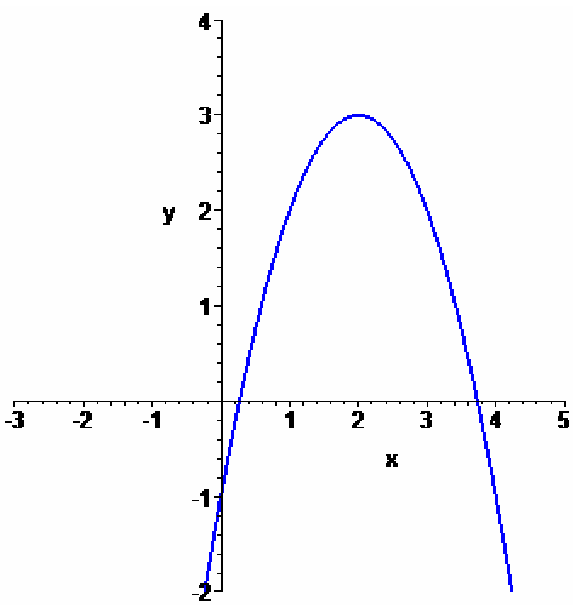
∎

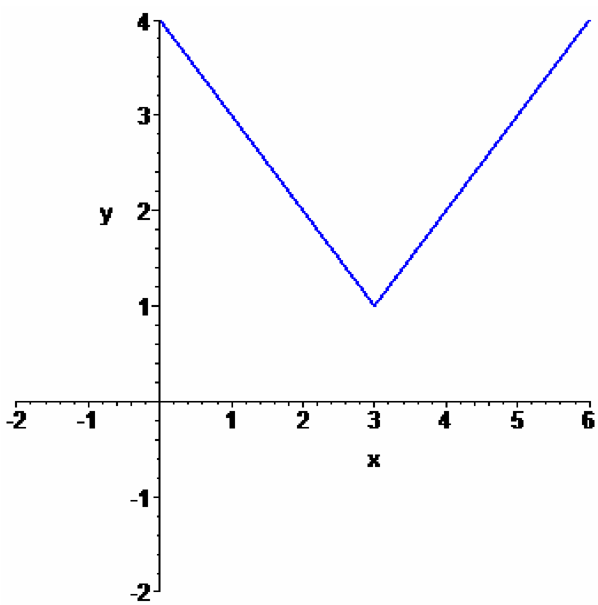
Note: Local maximum and local minimum points do not always give absolute maximum and minimum points.



Critical Numbers

If a function f is defined at $x = c$ ($x = c$ is in the domain of f), then $x = c$ is a critical number (critical point) if $f'(c) = 0$ or if $f'(c)$ is undefined.





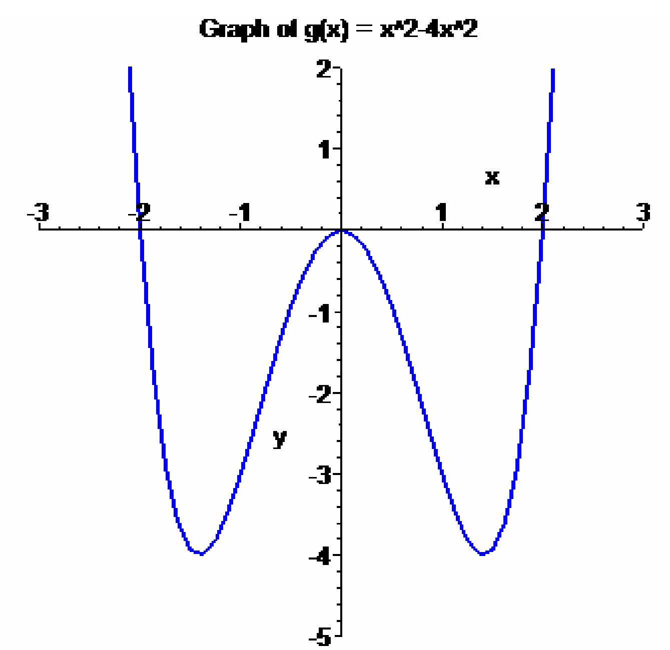
Fact: If f has a relative minimum or a relative maximum at $x = c$, then $x = c$ must be a critical number for the function f.

Note: Before determining the critical numbers for a function, you should state the domain of the function first.

Example 3: Find the critical numbers of the function $g(x) = x^4 - 4x^2$.

.

Solution:

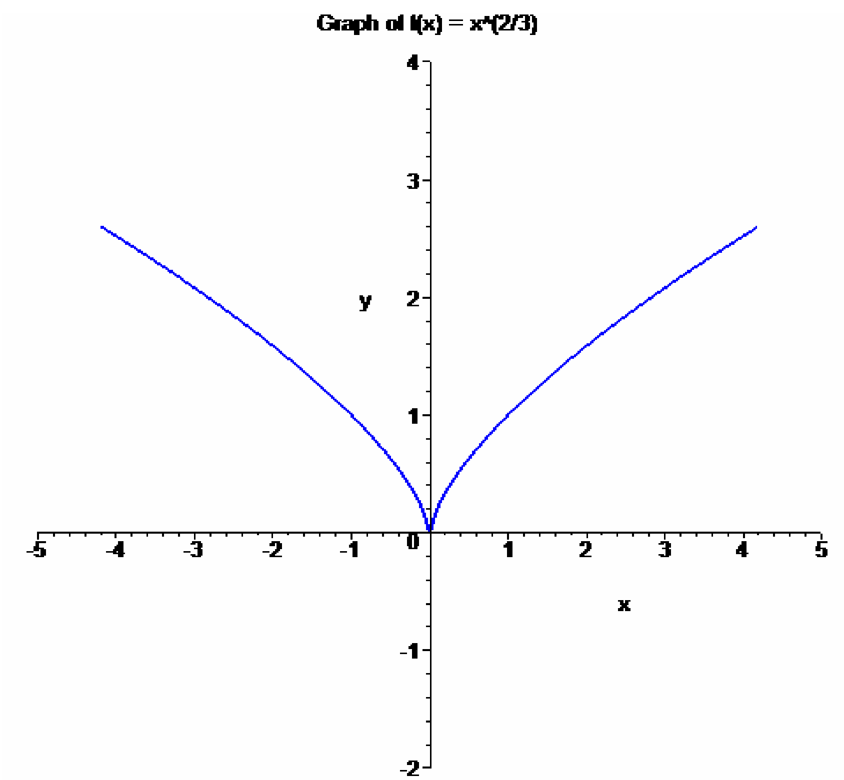


∎

Example 4: Find the critical numbers of the function $f(x) = x^{\frac{2}{3}}$.

.

Solution:



Note: Having $x = c$ be a critical number, that is, when $f'(c) = 0$ or $f'(c)$ is undefined, does not guarantee that $x = c$ produces a local maximum or local minimum for the function f.

Example 5: Demonstrate that the function $f(x) = x^3$ has a critical number but no local maximum or minimum point.

Solution:

■

The Extreme Value Theorem

If a function f is continuous on a closed interval $[a, b]$, then f has both an absolute minimum and an absolute maximum in $[a, b]$.

Steps for Evaluation Absolute Extrema on a Closed Interval

To find the absolute maximum and absolute minimum points for a continuous function f on the closed interval $[a, b]$.

1. Find the critical numbers of f (values of x where $f'(x) = 0$ or $f'(x)$ is undefined) that are contained in $[a, b]$. **Important!** You must make sure you only consider critical numbers for step 2 that are in $[a, b]$. For critical numbers not in $[a, b]$, you must throw these out and not consider them for step 2.
2. Evaluate f (find the y values) at each critical number in $[a, b]$ and at the endpoints of the interval $x = a$ and $x = b$.
3. The smallest of these values (smallest y value) from step 2 is the absolute minimum. The largest of these values (largest y value) is the absolute maximum.

Example 6: Find the absolute maximum and absolute minimum values for the function $f(x) = x^3 - 3x + 1$ on the interval [0, 3].

Solution:

Example 7: Find the absolute maximum and absolute minimum values for the function $f(x) = x \ln x$ on the interval [1, 4].

Solution:

Example 8: Find the absolute maximum and absolute minimum values for the function $f(x) = x - 2\cos x$ on the interval $[\frac{\pi}{2}, \frac{3\pi}{2}]$.

Solution: To find the candidates for the absolute maximum and minimum points, we first find the critical numbers. We first compute $f'(x) = 1 - 2(-\sin x) = 1 + 2\sin x$. Noting that the derivative f' is defined for all values of *x*, we then find the critical numbers by looking for values of *x* where $f'(x) = 0$. Hence, we set $f'(x) = 1 + 2\sin x = 0$ and solve for *x*. This gives

$$
\begin{aligned}
f'(x) = 1 + 2\sin x &= 0 \\
2\sin x &= -1 \qquad \text{(Subtract 1 from both sides)} \\
\sin x &= -\frac{1}{2} \qquad \text{(Divide by 2)}
\end{aligned}
$$

Within the interval $[\frac{\pi}{2}, \frac{3\pi}{2}]$, the sine function is negative in the third quadrant. Hence, the critical number within $[\frac{\pi}{2}, \frac{3\pi}{2}]$ where $\sin x = -\frac{1}{2}$ is $x = \frac{7\pi}{6}$. Thus the candidates for finding the absolute maximum and minimum points are the following:

$x = \frac{7\pi}{6}$ (critical number in the given interval $[\frac{\pi}{2}, \frac{3\pi}{2}]$)

$x = \frac{\pi}{2}, \frac{3\pi}{2}$ (endpoints of the interval $[\frac{\pi}{2}, \frac{3\pi}{2}]$)

We test these candidates using the <u>original</u> <u>function</u> $f(x) = x - 2\cos x$ to see which as the smallest and largest functional value (y-value). We see that

$$x = \frac{\pi}{2} \Rightarrow y = f(\frac{\pi}{2}) = \frac{\pi}{2} - 2\cos\frac{\pi}{2} = \frac{\pi}{2} - 2(0) = \frac{\pi}{2} - 0 = \frac{\pi}{2} \approx 1.57 \Leftarrow \text{Absolute Minimum.}$$

$$x = \frac{7\pi}{6} \Rightarrow y = f(\frac{7\pi}{6}) = \frac{7\pi}{6} - 2\cos\frac{7\pi}{6} = \frac{7\pi}{6} - 2(-\frac{\sqrt{3}}{2}) = \frac{7\pi}{6} + \sqrt{3} \approx 5.40 \Leftarrow \text{Absolute Maximum}$$

$$x = \frac{3\pi}{2} \Rightarrow y = f(\frac{3\pi}{2}) = \frac{3\pi}{2} - 2\cos\frac{3\pi}{2} = \frac{3\pi}{2} - 2(0) = \frac{3\pi}{2} - 0 = \frac{3\pi}{2} \approx 4.71$$

Hence, we see that the absolute extrema are the following:

Absolute maximum: $f(\frac{7\pi}{6}) = \frac{7\pi}{6} + \sqrt{3} \approx 5.40$	Absolute minimum: $f(\frac{\pi}{2}) = \frac{\pi}{2} \approx 1.57$