

## Section 4.3: Derivatives and the Shapes of Curves

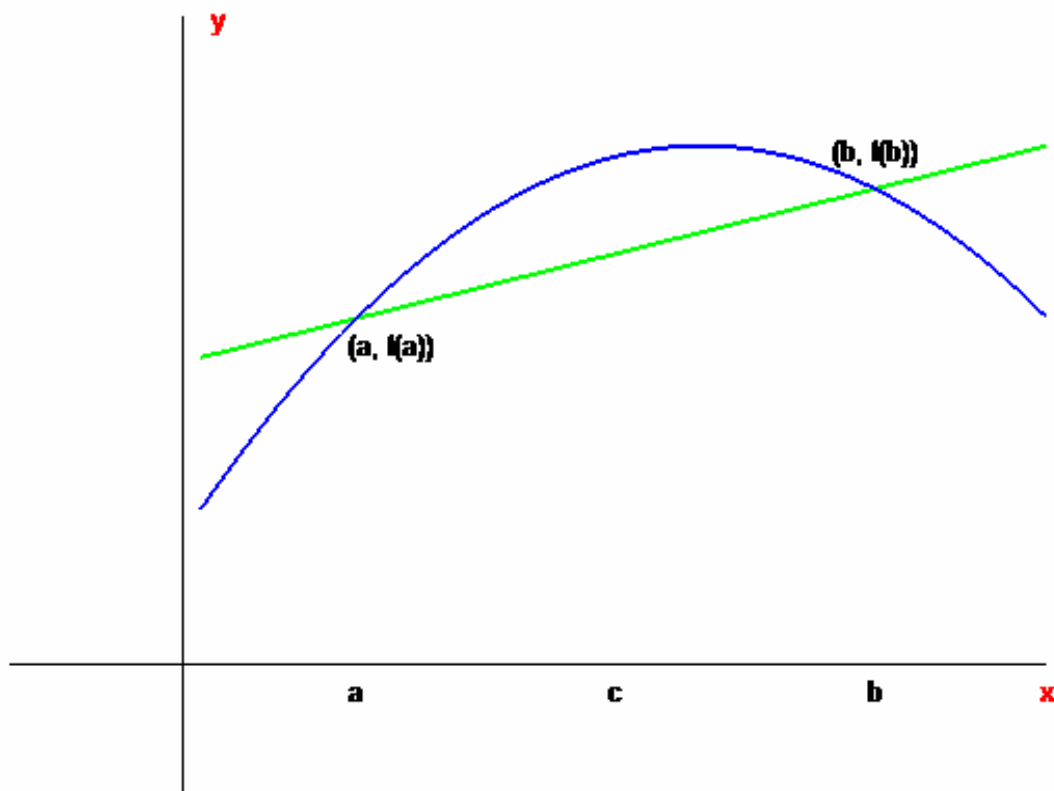
Practice HW from Stewart Textbook (not to hand in)  
p. 286 # 1, 3, 7, 9, 11, 19, 21, 23, 25 odd

### The Mean Value Theorem

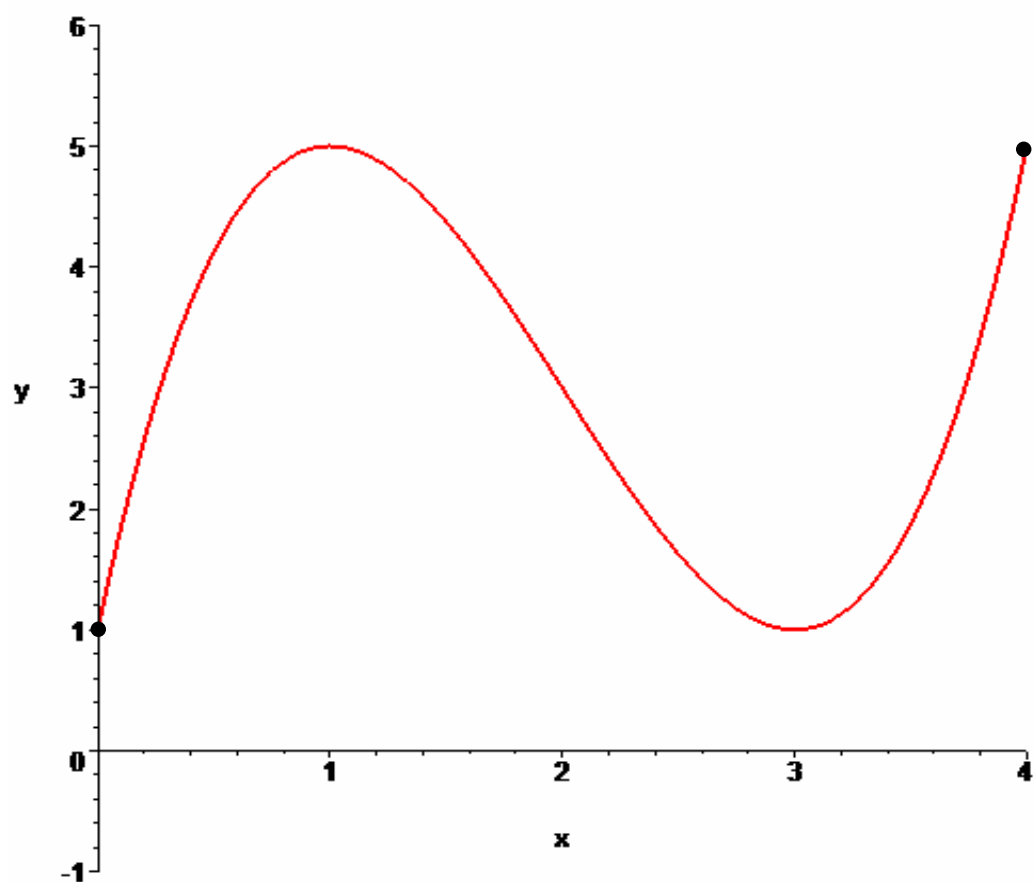
If  $f$  is a continuous function on the closed interval  $[a, b]$  and differentiable on the open interval  $(a, b)$ , then there exists a number  $x = c$  in  $(a, b)$ , that is, a number  $x = c$  where  $a < c < b$ , such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Geometric Interpretation of the Mean Value Theorem:



**Example 1:** Use the graph of the following function to estimate a value of  $c$  that satisfies the conclusion of the Mean Value Theorem for the interval  $[0, 4]$ .



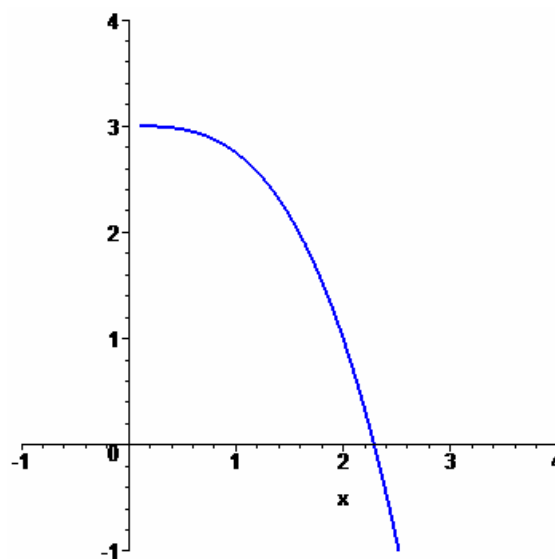
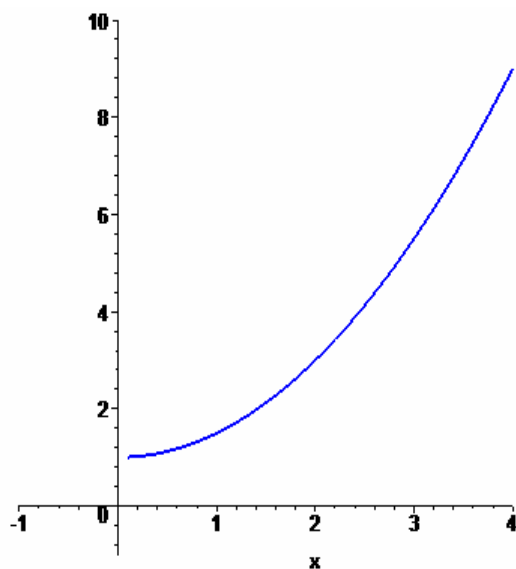
**Solution:**



## Increasing and Decreasing Test for Functions

Given a function  $f$  defined on an interval.

1. If  $f'(x) > 0$  on an interval, then  $f$  is increasing on that interval.
2. If  $f'(x) < 0$  on an interval, then  $f$  is decreasing on that interval.



**Example 2:** State whether the function  $f(x) = x^3 - 3x$  is increasing, decreasing, or neither on an interval “near”  $x = 2$ .

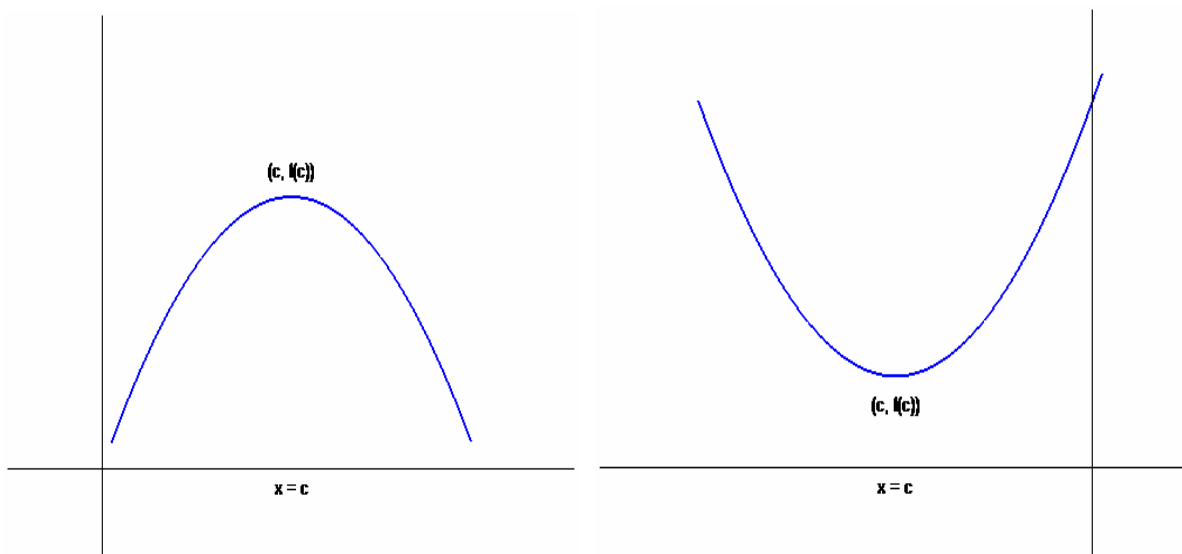
**Solution:**



## First Derivative Test

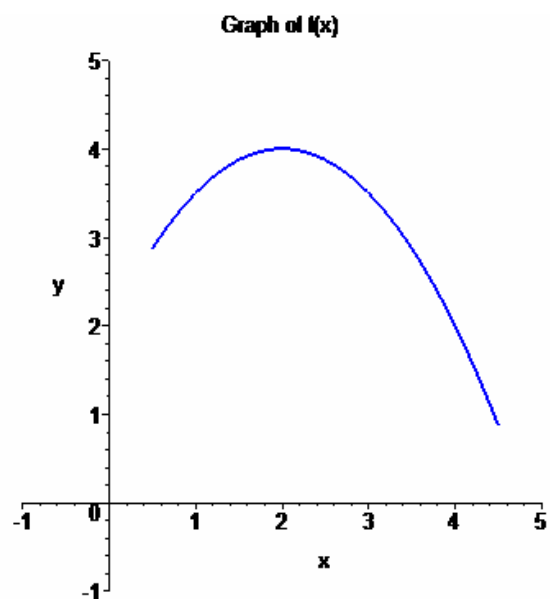
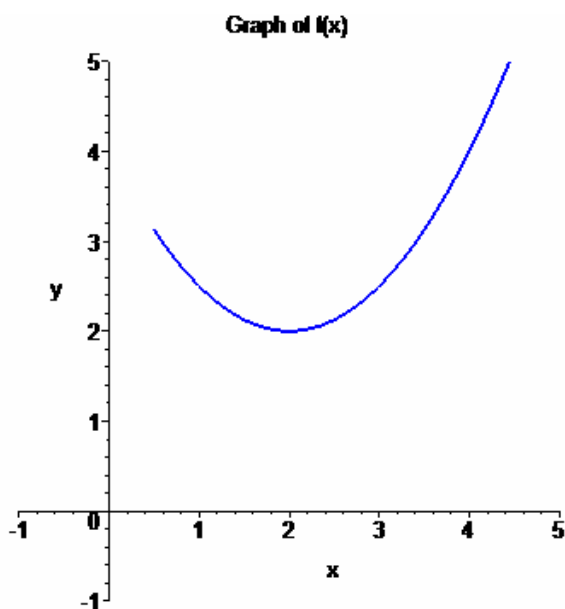
Suppose  $x = c$  is a critical number of a continuous function  $f$ .

1. If  $f'(x) > 0$  for  $x < c$  and  $f'(x) < 0$  for  $x > c$ ,  $f(c)$  is a relative maximum.
2. If  $f'(x) < 0$  for  $x < c$  and  $f'(x) > 0$  for  $x > c$ ,  $f(c)$  is a relative minimum.



## Concavity

- A function  $f$  is concave up on an interval  $I$  if  $f'$  is increasing on  $I$ .
- A function  $f$  is concave down on an interval  $I$  if  $f'$  is decreasing on  $I$ .



## Concavity Test for Functions

Given a function  $f$  defined on an interval.

1. If  $f''(x) > 0$  on an interval, then  $f$  is concave up on that interval.
2. If  $f''(x) < 0$  on an interval, then  $f$  is concave down on that interval.

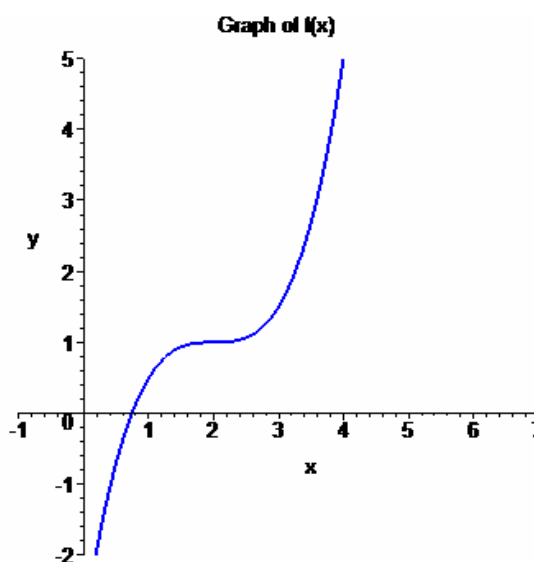
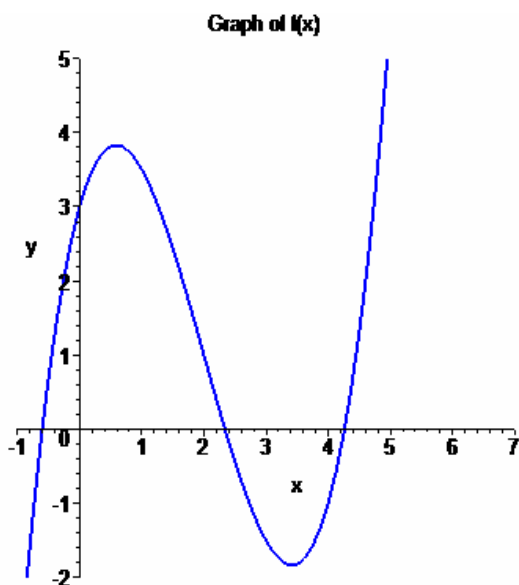
**Example 2:** State whether the function  $f(x) = x^3 - 3x$  is concave up, concave down, or neither on an interval “near”  $x = -1$ .

**Solution:**



## Inflection Points

*Inflection points* are points where the concavity of the graph of a function  $f$  changes.

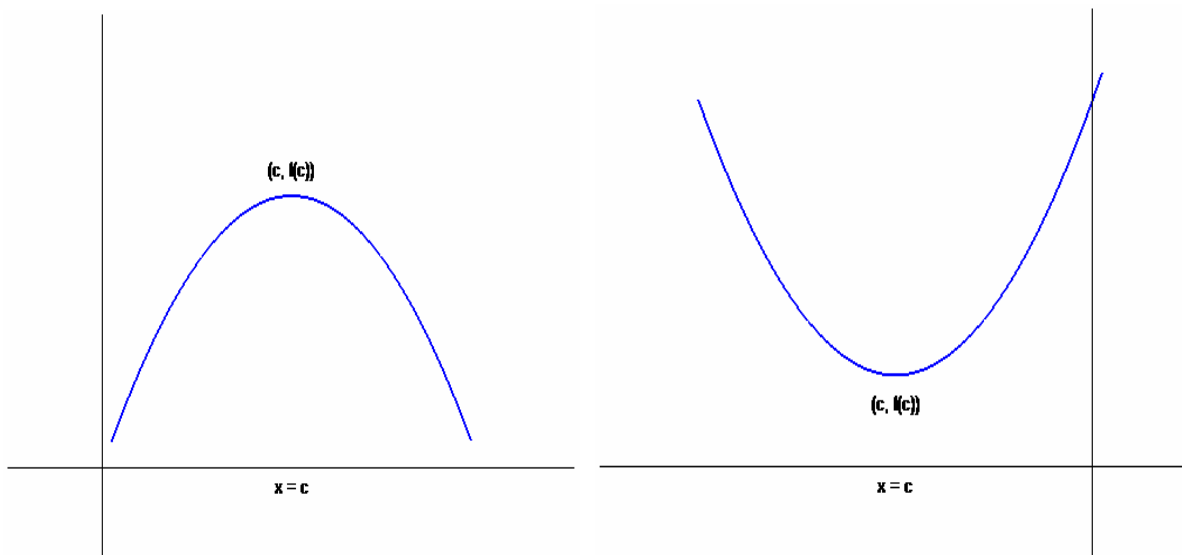


**Note:** If  $(c, f(c))$  is a point of inflection of the graph of  $f$ , then either  $f''(c) = 0$  or  $f''(c)$  is undefined.

## Second Derivative Test (Test For Relative Maximum and Relative Minimum Points)

Let  $f$  be a function where  $x = c$  be a critical point where  $f'(c) = 0$ .

1. If  $f''(c) > 0$ , then  $f(c)$  is a relative minimum.
2. If  $f''(c) < 0$ , then  $f(c)$  is a relative maximum.
3. If  $f''(c) = 0$  or  $f''(c)$  is undefined, the test fails – use the 1<sup>st</sup> derivative test.



## Procedure For Graphing Functions Using Derivatives

Given a function  $f(x)$ .

1. State the domain of the function.
2. Compute  $f'$  and  $f''$ .
3. Find the critical numbers, which are values of  $x$  where either  $f'(x) = 0$  or  $f'(x)$  is undefined. Find the  $y$  coordinates of these points by substituting these  $x$  values back into the original function  $f(x)$ . These points represent the candidates for the local maximum and minimum points.
4. Use the 2<sup>nd</sup> derivative test to determine whether critical numbers are local maximum, local minimum points, or points of inflection. If the 2<sup>nd</sup> derivative test fails, use the 1<sup>st</sup> derivative test with a sign diagram, testing whether the 1<sup>st</sup> derivative is positive or negative to the left and right of the critical numbers.
5. Look for candidates for the points of inflection by finding values of  $x$  where either  $f''(x) = 0$  or  $f''(x)$  is undefined. Find the  $y$  coordinates of these points by substituting these  $x$  values back into the original function  $f(x)$ . These points represent the candidates for the points of inflection.
6. Test the inflection point candidates by testing the concavity to the left and the right of inflection point candidates using a sign diagram with the 2<sup>nd</sup> derivative.
7. Use the information to sketch the graph.

**Example 4:** For the function  $f(x) = x^3 - 3x^2 + 1$ , determine the intervals where the function is increasing and decreasing, local maximum and local minimum points, the intervals where the function is concave up and concave down, and the points of inflection. Use the information to sketch the graph.

**Solution:**





**Example 5:** For the function  $f(x) = x^4 - 4x^3$ , determine the intervals where the function is increasing and decreasing, local maximum and local minimum points, the intervals where the function is concave up and concave down, and the points of inflection. Use the information to sketch the graph.



**Example 6:** For the function  $f(x) = 3x^{2/3} - x$ , determine the intervals where the function is increasing and decreasing, local maximum and local minimum points, the intervals where the function is concave up and concave down, and the points of inflection. Use the information to sketch the graph.

**Solution:** Note that for this function  $f(x) = 3x^{2/3} - x$ , there is no values of  $x$  where the  $f(x)$  is undefined. Hence, the domain of this function is all real numbers or  $(-\infty, \infty)$ . We next compute the first and second derivatives. We see that

$$f'(x) = 3\left(\frac{2}{3}\right)x^{-1/3} - 1 = 2x^{-1/3} - 1 = \frac{2}{x^{1/3}} - 1 = \frac{2}{\sqrt[3]{x}} - 1.$$

Using the form  $f'(x) = 2x^{-1/3} - 1$  for the first derivative, we see that

$$f''(x) = 2\left(-\frac{1}{3}\right)x^{-4/3} - 0 = -\frac{2}{3x^{4/3}} = \frac{-2}{3\sqrt[3]{x^4}}.$$

$$\text{Hence, } f'(x) = \frac{2}{\sqrt[3]{x}} - 1 \text{ and } f''(x) = \frac{-2}{3\sqrt[3]{x^4}}.$$

We next find the critical numbers. Note that in this case,  $f'(x) = \frac{2}{\sqrt[3]{x}} - 1$  is undefined at  $x = 0$ . Therefore,  $x = 0$  is one critical number. We next look for critical numbers where  $f'(x) = 0$ . This gives

$$f'(x) = \frac{2}{\sqrt[3]{x}} - 1 = 0$$

$$\frac{2}{\sqrt[3]{x}} = 1 \quad (\text{add 1 to both sides})$$

$$\sqrt[3]{x} \cdot \frac{2}{\sqrt[3]{x}} = \sqrt[3]{x} \cdot 1 \quad (\text{Multiply both sides by } \sqrt[3]{x})$$

$$\sqrt[3]{x} = 2 \quad (\text{Rearrange and simplify})$$

$$(\sqrt[3]{x})^3 = (2)^3 \quad (\text{Eliminate the radical by raising both sides to the 3rd power})$$

$$x = 8 \quad (\text{Simplify and solve for } x)$$

Hence, the critical numbers are  $x = 0$  and  $x = 8$ . **(continued on next page)**

For the critical numbers, we find their corresponding  $y$  coordinates. To do this, we must use the original function  $f(x) = 3x^{2/3} - x$ . This gives the following.

$$x = 0 \Rightarrow y = f(0) = 3(0)^{2/3} - 0 = 0 - 0 = 0 \text{ gives the point } (0, 0)$$

$$x = 8 \Rightarrow y = f(8) = 3(8)^{2/3} - 8 = 3[8^{\frac{1}{3}}]^2 - 8 = 3[\sqrt[3]{8}]^2 - 8 = 3(2)^2 - 8 = 3(4) - 8 = 12 - 8 = 4$$

gives the point  $(8, 4)$

Hence,  $(0, 0)$  and  $(8, 4)$  are the local maximum and minimum point candidates.

To test these points, we first try the 2<sup>nd</sup> derivative test using  $f''(x) = \frac{-2}{3\sqrt[3]{x^4}}$ . This gives the following information for each candidate.

$$(0,0) \Rightarrow x = 0 \Rightarrow f''(0) = \frac{-2}{3\sqrt[3]{0^4}} = -\frac{2}{0} \text{ which is undefined. The test fails.}$$

$$(8,4) \Rightarrow x = 8 \Rightarrow f''(8) = \frac{-2}{3\sqrt[3]{8^4}} = -\frac{2}{3\sqrt[3]{4096}} = -\frac{2}{3(16)} = -\frac{1}{24} < 0. \text{ This says the graph is concave down at this point or that } (8, 4) \text{ is a local maximum.}$$

Hence, the 2<sup>nd</sup> derivative test says  $(8, 4)$  is a local maximum but is inconclusive for the point  $(0, 0)$ . So we need more information. We get it by using the 1<sup>st</sup> derivative test by testing the sign of the 1<sup>st</sup> derivative to the left and right of each critical number (which of course are the same as the  $x$  coordinates of the local maximum and minimum candidates).

1<sup>st</sup> Derivative Test Table for  $f'(x) = \frac{2}{\sqrt[3]{x}} - 1$

$f'(x) < 0$ ----- Decreasing	<b>Crit Num <math>x = 0</math></b>	$f'(x) > 0$ +++++ Increasing	<b>Crit Num <math>x = 8</math></b>	$f'(x) < 0$ ----- Decreasing
<b>Choose <math>x = -1 &lt; 0</math></b> $f'(-1) = \frac{2}{\sqrt[3]{-1}} - 1$ $= \frac{2}{-1} - 1$ $= -2 - 1$ $= -3 < 0$ Decreasing		<b>Choose <math>x = 1</math></b> <b>Note <math>0 &lt; 1 &lt; 8</math></b> $f'(1) = \frac{2}{\sqrt[3]{1}} - 1$ $= \frac{2}{1} - 1$ $= 2 - 1$ $= 1 > 0$ Increasing		<b>Choose <math>x = 9 &gt; 8</math></b> $f'(9) = \frac{2}{\sqrt[3]{9}} - 1$ $= \frac{2}{9^{1/3}} - 1$ $\approx \frac{2}{2.08} - 1$ $= 0.96 - 1$ $= -0.04 < 0$ Decreasing

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To the left of the critical number  $x = 0$ , the 1<sup>st</sup> derivative test table says the graph of  $f(x)$  is decreasing. To the right of  $x = 0$ , the 1<sup>st</sup> derivative test table says the graph of  $f(x)$  is increasing. By definition, this says the point  $(0, 0)$  is a local minimum.

To the left of the critical number  $x = 8$ , the 1<sup>st</sup> derivative test table says the graph of  $f(x)$  is increasing. To the right of  $x = 8$ , the 1<sup>st</sup> derivative test table says the graph of  $f(x)$  is decreasing. By definition, this says the point  $(8, 4)$  is a local maximum (which agrees with the assertion already established by the 2<sup>nd</sup> Derivative test).

The table also gives information where the graph of  $f(x)$  is increasing and decreasing. Summarizing, we see that

$f(x)$  is increasing when  $0 < x < 8$  or in interval notation  $(0, 8)$

$f(x)$  is decreasing when  $x < 0, x > 8$  or in interval notation  $(-\infty, 0) \cup (8, \infty)$

Now, we look for points of inflection. Recall that points of inflection occur at points where the second derivative  $f''(x) = 0$  or where  $f''(x)$  is undefined. Above, we

computed the second derivative  $f''(x) = \frac{-2}{3\sqrt[3]{x^4}}$ . Note that  $f''(x) = \frac{-2}{3\sqrt[3]{x^4}}$  is undefined at

$x = 0$ . Thus  $x = 0$  is inflection point candidate. If we set

$$f''(x) = \frac{-2}{3\sqrt[3]{x^4}} = 0$$

$$3\sqrt[3]{x^4} \cdot \frac{-2}{3\sqrt[3]{x^4}} = 3\sqrt[3]{x^4} \cdot 0 \quad (\text{Multiply both sides by } 3\sqrt[3]{x^4} \text{ to eliminate the fraction})$$

$$-2 = 0 \quad (\text{Simplify})$$

Since  $-2 \neq 0$ , this produces not solution. Thus,  $x = 0$  is the only inflection point candidate. Recall that its y-coordinate on the graph is obtain by substituting into the original function  $f(x) = 3x^{2/3} - x$ . Recall that this gives the following.

$$x = 0 \Rightarrow y = f(0) = 3(0)^{2/3} - 0 = 0 - 0 = 0 \text{ gives the point } (0, 0)$$

To test whether the concavity changes, we use a test table involving the second derivative.

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2<sup>nd</sup> Derivative Test Table for  $f''(x) = \frac{-2}{3\sqrt[3]{x^4}}$

$f''(x) < 0$ ----- Concave Down	Inflection point Candidate $x = 0$	$f''(x) < 0$ ----- Concave Down
<b>Choose</b> $x = -1 < 0$ $f''(-1) = \frac{-2}{3\sqrt[3]{(-1)^4}}$ $= \frac{-2}{3\sqrt[3]{1}}$ $= \frac{-2}{3(1)}$ $= -\frac{2}{3} < 0$ Concave Down		<b>Choose</b> $x = 1 > 0$ $f''(1) = \frac{-2}{3\sqrt[3]{(1)^4}}$ $= \frac{-2}{3\sqrt[3]{1}}$ $= \frac{-2}{3(1)}$ $= -\frac{2}{3} < 0$ Concave Down

Since the concavity of the graph does not change to the left and right of  $x = 0$  (it remains concave down), the point  $(0, 0)$  does not produce a point of inflection. Since there are no other candidates, the graph of  $f(x)$  has no points of inflection. Using the table, we see that we have the following intervals for concavity.

Concave Down: When  $x < 0$ ,  $x > 0$  or in interval notation  $(-\infty, 0) \cup (0, \infty)$ .

Concave Up: Never

In summary, this is the information we have produced:

1. The point  $(0, 0)$  is a local minimum, the point  $(8, 4)$  is a local maximum.
2.  $f(x)$  is increasing on the interval  $(0, 8)$ .  
 $f(x)$  is decreasing on the interval  $(-\infty, 0) \cup (8, \infty)$ .
3. There are no points of inflection.
4. The graph is concave down on the interval  $(-\infty, 0) \cup (0, \infty)$ .  
The graph is never concave up.
5. Note that the local minimum,  $(0, 0)$ , was obtained from an undefined critical point, that is,  $f'(x) = \frac{2}{\sqrt[3]{x}} - 1$  was undefined at  $x = 0$ . A function has an undefined derivative at a point where the graph is not continuous, has a sharp point, or a vertical tangent. The function  $f(x) = 3x^{2/3} - x$  is continuous at  $x = 0$ . Since  $(0, 0)$  is a local minimum, the graph changes from decreasing to increasing at this point. In addition, the concavity remains down to the left and right of this point. Hence, the graph produces a sharp point at  $(0, 0)$ .

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The following is the graph of the function  $f(x) = 3x^{2/3} - x$ .

