Section 4.8: Newton's Method

Practice HW from Stewart Textbook (not to hand in)
p. 325 # 1, 5-15 odd, 23

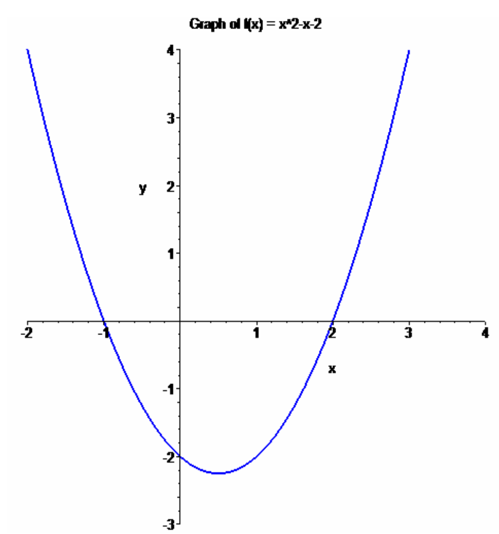
Newton's Method

Recall: A root (zero) of a function $y = f(x)$ is a value $x = c$ where $y = f(c) = 0$.

Example 1: Find the zeros of the function $f(x) = x^2 - x - 2$. Verify your answer is correct.

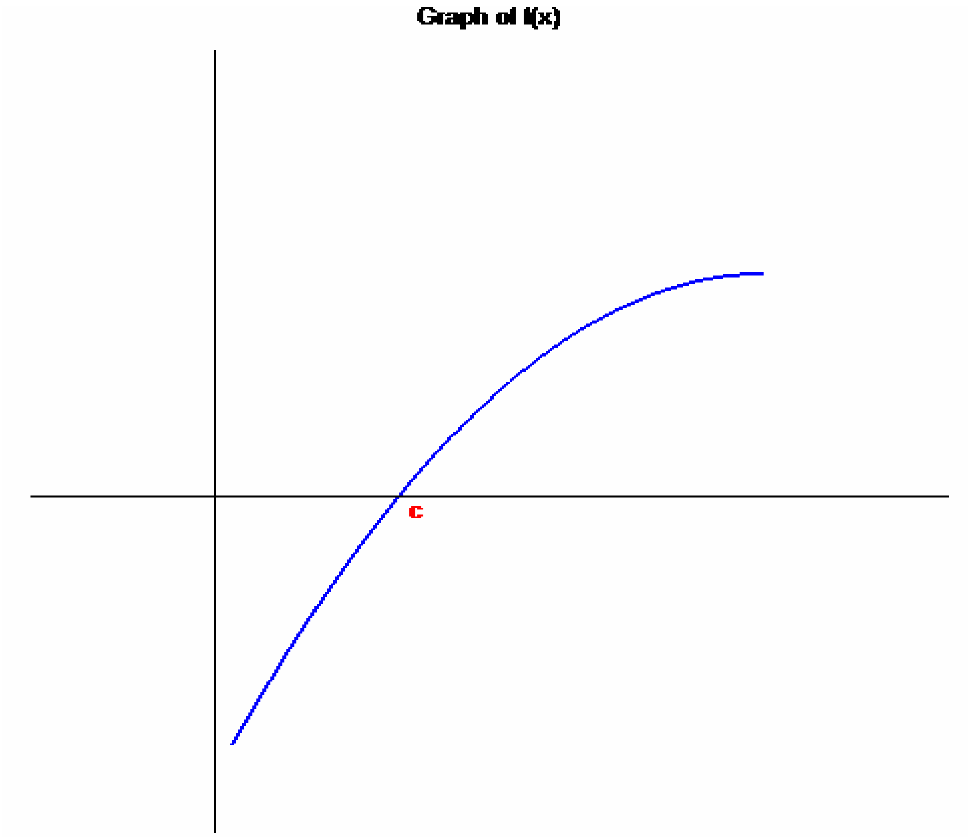
Solution:

Fact: Graphically, the roots (zeros) give the *x*-intercepts of a function.



In this section, we need a method to approximate zeros, which are not easily found by algebraic methods.

Suppose we want to find a root $x = c$ of the following function $y = f(x)$.



We make a guess x_1 which is "sufficiently" close to the solution. We use the point slope equation to find equation of the tangent line at the point $(x_1, f(x_1))$.

$$y - y_1 = m_{\tan}(x - x_1)$$

$$y - f(x_1) = f'(x_1)(x - x_1)$$

$$y = f'(x_1)(x - x_1) + f(x_1)$$

Consider the point where the graph of the tangent line intercepts the x axis, which we label as $x = x_2$. The idea is the value of x_2 is "closer" to the zero $x = c$ than x_1. At $x = x_2$, $y = 0$ and we have the point $(x_2, 0)$. This gives the equation

$$0 = f'(x_1)(x_2 - x_1) + f(x_1)$$

We solve this equation for x_2.

$$f'(x_1)(x_2 - x_1) = -f(x_1)$$

$$x_2 - x_1 = \frac{f(x_1)}{f'(x_1)}$$

$$x_2 = x_1 + \frac{f(x_1)}{f'(x_1)}$$

We next find the equation of the tangent line to the graph of $f(x)$ at the point $(x_2, f(x_2))$. This gives

$$y - f(x_2) = f'(x_2)(x - x_2)$$

Suppose this tangent line intercepts the x axis at the point $(x_3, 0)$. Using a similar process described above, we find that x_3 is given by

$$x_3 = x_2 + \frac{f(x_2)}{f'(x_2)}$$

The value $x = x_3$ is even closer to the zero of $x = c$ than $x = x_2$. Generalizing this idea gives us the following algorithm.

Newton's Method (Newton Raphson Method)

Given an initial value x_1 that is sufficiently "close" to a zero $x = c$ of a function $f(x)$. Then the following iterative calculation

$$x_{n+1} = x_n + \frac{f(x_n)}{f'(x_n)}$$

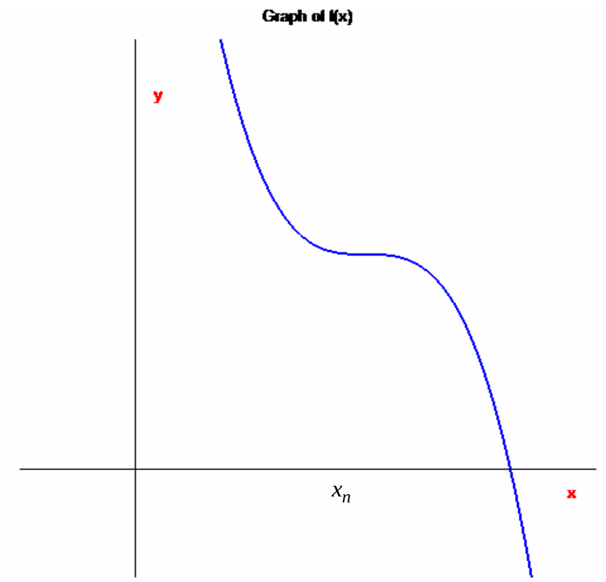
can be used to successively better approximate the zero $x = c$.

Notes:

1. We usually generate iterations until two successive iterates agree to a specified number of decimal places.
2. Newton's method generally converges to a zero in a small number (< 10) iterations.
3. Newton's method may fail in certain instances.

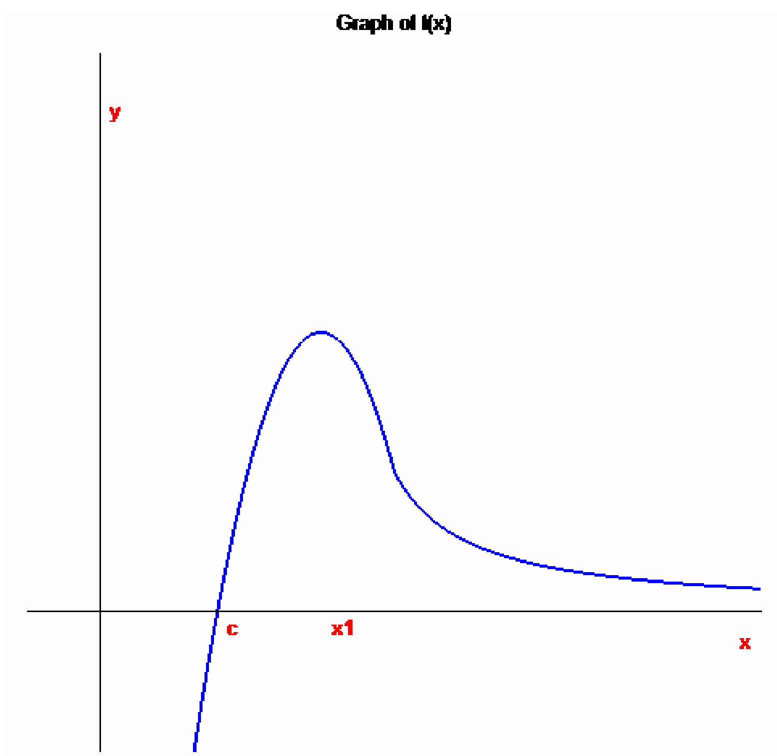
Examples of Failure for Newton's Method

1. Iterate produces a tangent line slope of zero.



$$x_{n+1} = x_n + \left.\frac{f(x_n)}{f'(x_n)}\right\} \text{undefined since } f'(x_n) = 0$$

2. Initial guess x_1 not close enough



Example 2: Use Newton's Method to approximate the root of $x^5 + x - 1 = 0$ to six decimal places.

Solution:

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Example 3: Use Newton's Method to approximate the solution of the equation $(x-1)^3 = \sin x$ to eight decimal places.

Solution: