Section 4.9: Antiderivatives

Practice HW from Stewart Textbook (not to hand in)
p. 332 # 1-27 odd, 39, 40, 41

Antidifferentiation or Integration

Suppose we are given a derivative of a function $f'(x) = 3x^2$ and asked to find $f(x)$.

There are many answers for $f(x)$ such as:

In general, we say that

$$f(x) = x^3 + C$$

where C is known as the *constant of integration.*

Antidifferentiation or integration is the opposite of differentiation.

Notation: We use the indefinite integral to denote the antiderivative.

$$\int f(x)\ dx$$

Thus, $\int 3x^2\ dx = x^3 + C$.

Basic Antiderivative (Integration Formulas) p. 329

1. $\int k\ dx =$

   $\int dx =$

   $\int 0\ dx =$

2. $\int x^n\ dx =$

3. $\int x^{-1}\ dx =$

4. $\int \sin x\ \ dx =$

5. $\int \cos x\ \ dx =$

6. $\int \sec^2 x\ \ dx =$

7. $\int \sec x\ \ \tan x\ \ dx =$

8. $\int \csc^2 x\ \ dx =$

9. $\int \csc x\ \ \cot x\ \ dx =$

Example 1: Find the antiderivative of the function $f(x) = x^3$.

Solution:

Example 2: Find the antiderivative of the function $f(t) = \sqrt{t}$.

Solution:

∎

Example 3: Find the antiderivative of the function $f(x) = \dfrac{1}{x^7}$.

Solution:

∎

Example 4: Find the antiderivative of the function $f(x) = 3$.

Solution:

Properties of Integration

1. $\int k\ f(x)\,dx = k\int f(x)\,dx$, where k is a constant – in our case a real number)
2. $\int \big(f(x) \pm g(x)\big)\,dx = \int f(x)\,dx \pm \int g(x)\,dx$

Example 5: Find the antiderivative of the function $f(x) = 2x^2 - 3x + 3$.

Solution:

Example 6: Find the antiderivative of the function $f(t) = t - 2\sin t + 3\cos t$.

Solution:

$\blacksquare$

Example 7: Find the antiderivative of the function $f(x) = \frac{3}{x^4} + \frac{3}{x} + x^{\frac{1}{4}} + \sec^2 x$.

Solution:

$\blacksquare$

Differential Equations

Differential equations are equations involving one or more of its derivatives. A simple example of a differential equation is given by

$$f'(x) = 3x^2$$

To find $f(x)$, we integrate both sides with respect to x.

$$\int f'(x)\,dx = \int 3x^2\,dx$$

which gives

$$f(x) = x^3 + C \leftarrow \text{known as the general solution}$$

The *general solution* expresses the solution in terms of the arbitrary constant C. If we are given an *initial condition* (a value for the function at a particular value of x), we can find the *particular solution* (where we find a particular value for the integration constant C).

Example 8: Solve the differential equation $f'(x) = 3x^2$ when $f(0) = 2$.

Solution:

■

Example 9: Find f given $f''(x) = \sin x$ where $f'(0) = 1$, $f(0) = 6$

Solution:

■

Vertical Motion

Recall that given a position function $s(t)$.

Velocity: $v(t) = s'(t)$
Acceleration: $a(t) = v'(t) = s''(t)$

Hence, since integration is the opposite of differentiation, we can say:

Velocity: $v(t) = \int a(t)\, dt$
Acceleration: $s(t) = \int v(t)\, dt$

Example 10: A ball is thrown vertically upward from the ground at an initial height of 5 ft with an initial velocity of 64 ft/s.

a. Find the position function $s(t)$

b. How high will the ball go?

c. How long thus it take for the ball to hit the ground.

Solution: Part a. In this problem, we start with the fact that the acceleration due to gravity of a freely falling object is – 32 ft/s (-9.8 m/s in metric). Thus we can say that the acceleration equation is given by

$$a(t) = -32$$

Since acceleration is the derivative of the velocity, we must reverse the process and integrate the acceleration to get the velocity. This gives the following.

$$v(t) = \int a(t)\, dt$$

$$v(t) = \int -32\, dt$$

$$v(t) = -32\, t + C$$

To find the constant *C*, we can use the fact that the initial velocity (the velocity at time $t = 0$) is 64 ft/s, which translates mathematically as $v(0) = 64$. Substituting into the velocity equation gives

$$\begin{aligned} 64 = v(0) &= -32(0) + C \\ 64 &= 0 + C \\ C &= 64 \end{aligned}$$

Thus, we see that the velocity equation is $v(t) = -32\, t + 64$. To find the position function $s(t)$, we use the fact the derivative of the position gives the velocity. Hence, we must integrate the velocity to get the position. Hence, we have

$$s(t) = \int v(t)\, dt$$

$$s(t) = \int (-32t + 64)\, dt$$

$$s(t) = -32\frac{t^2}{2} + 64t + D$$

$$s(t) = -16t^2 + 64t + D$$

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To find the constant D, we use the fact that the initial height (initial position) of the height at time $t = 0$ is 5 ft. Mathematically, this translates as $s(0) = 5$. Using this condition, we obtain

$$5 = s(0) = -16(0)^2 + 64(0) + D$$
$$5 = 0 + 0 + D$$
$$D = 5$$

Thus the position function for the ball height is $\boxed{s(t) = -16t^2 + 64t + 5.}$

Part b. The ball reaches its maximum height at the time when the velocity $v(t) = 0$. Hence we take the velocity equation we found in part a, set it equal to 0, and solve for t. This gives

$$v(t) = -32t + 64 = 0$$
$$-32t = -64$$
$$t = \frac{-64}{-32} = 2.$$

Thus, the object reaches its maximum height after t = 2 seconds. To find its height at this time, we simply substitute this value of t into the position equation. This gives

$$\text{Height of ball} = s(2) = -16(2)^2 + 64(2) + 5 = -64 + 128 + 5 = 69 \text{ ft}$$

This, the ball goes 69 ft high.

Part c. In this problem, we want to find the time t it takes for the ball to go up, come back down, and hit the ground. When the ball hits the ground, its height is 0. Thus, we can find t by setting the height function $s(t) = -16t^2 + 64t + 5$ equal to 0 and solving for t. This gives

$$s(t) = -16t^2 + 64t + 5 = 0$$

We use the quadratic formula to solve this equation. Recall that this formula says that the solutions to the quadratic equation $ax^2 + bx + c = 0$ is $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

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If we replace *x* with *t* and assign *a* = -16, *b* = 64, and *c* = 5, we see that

$$t = \frac{-64 \pm \sqrt{(64)^2 - 4(-16)(5)}}{2(-16)}.$$

$$t = \frac{-64 \pm \sqrt{4096 + 320}}{-32}$$

$$t = \frac{-64 \pm \sqrt{4416}}{-32}$$

$$t \approx \frac{-64 \pm 66.5}{-32}$$

$$t = \frac{-64 - 66.5}{-32}, t = \frac{-64 + 66.5}{-32}$$

$$t = \frac{-64 - 66.5}{-32}, t = \frac{-64 + 66.5}{-32}$$

$$t = \frac{-130.5}{-32}, \quad t = \frac{2.5}{-32}$$

$t \approx 4.1$, ~~$t \approx -0.08$~~

Thus, the ball will hit the ground after 4.1 seconds.