Section 5.10: Improper Integrals

Practice HW from Stewart Textbook (not to hand in)
p. 431 # 5-31 odd

Areas of Infinite Extent

Example 1: Determine the area under the graph of $f(x) = \dfrac{1}{x^2}$ for $x \geq 1$

Solution:

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The type of integral used to compute an area of infinite extent is called an <u>improper integral</u>.

Definition of Improper Integrals (Type 1)

1. $\int_a^{\infty} f(x)\,dx = \lim_{t\to\infty} \int_a^{t} f(x)\,dx$

2. $\int_{-\infty}^{b} f(x)\,dx = \lim_{t\to -\infty} \int_t^{b} f(x)\,dx$

3. $\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_a^{-\infty} f(x)\,dx$

If the limit of an improper integral exists (approaches a fixed number), then the improper integral is convergent. If the limit does not exist, then the improper integral is divergent.

Fact: Note that $\int_1^{\infty} \frac{1}{x^2}\,dx = 1$ is convergent.

Example 2: Compute $\int_1^{\infty} \frac{1}{x}\,dx$

Solution:

Example 3: Compute $\int_0^{\infty} e^{-2x}\, dx$.

Solution:

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Example 4: Compute $\int_{-\infty}^{0} e^{-2x}\, dx$.

Solution:

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Example 5: Compute $\int_{-\infty}^{\infty} x^3 e^{-x^4}\ dx$

Solution:

$\blacksquare$

Discontinuous Integrands

Looks at areas of infinite extent when *x* approaches a value where the function is undefined.

Example 6: Compute $\int_0^1 \frac{1}{\sqrt{x}}\,dx$

Solution:

$\blacksquare$

Example 7: Compute $\int_{-2}^{2} \frac{1}{(x-1)^3}\, dx$

Solution: The function $f(x) = \frac{1}{(x-1)^3}$ is undefined at x = 1. Hence, we want to examine what is happening to the integral near this point. We rewrite the integral as follows:

$$\int_{-2}^{2} \frac{1}{(x-1)^3}\, dx = \int_{-2}^{1} \frac{1}{(x-1)^3}\, dx + \int_{1}^{2} \frac{1}{(x-1)^3}\, dx$$

Hence, we must evaluate both of these integrals. We first evaluate $\int_{-2}^{1} \frac{1}{(x-1)^3}\, dx$. This

$$\int_{-2}^{1} \frac{1}{(x-1)^3}\, dx = \lim_{t \to 1^+} \int_{-2}^{t} \frac{1}{(x-1)^3}\, dx$$

To find $\int \frac{1}{(x-1)^3}\, dx$, we use a simple *u*-*du* substitute. We obtain the following:

$\int \frac{1}{(x-1)^3}\, dx$	Let $u = x - 1$
$= \int \frac{1}{u^3}\, du$	$\frac{du}{dx} = 1$
$= \int u^{-3} du$	$du = dx$
$= \frac{u^{-2}}{-2} + C$	
$= -\frac{1}{2u^2} + C$	
$= -\frac{1}{2(x-1)^2} + C$	

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$$\int_{-2}^{1} \frac{1}{(x-1)^3}\,dx = \lim_{t\to 1^+} \int_{-2}^{t} \frac{1}{(x-1)^3}\,dx$$

$$= \lim_{t\to 1^+} -\frac{1}{2(x-1)^2}\Bigg|_{-2}^{t}$$

Hence,

$$= \lim_{t\to 1^+}\left(-\frac{1}{2(t-1)^2} - -\frac{1}{2(-2-1)^2}\right)$$

$$= \lim_{t\to 1^+}\left(-\frac{1}{2(t-1)^2} + \frac{1}{2(-3)^2}\right)$$

$$= \lim_{t\to 1^+}\left(-\frac{1}{2(t-1)^2} + \frac{1}{18}\right)$$

As $t \to 1^+$, $-\frac{1}{2(t-1)^2} \to -\infty$. The following chart will help convince yourself of this.

t	$-\frac{1}{2(t-1)^2}$
2	-0.5
1.5	-2
1.1	-50
1.01	-5000
1.001	-500000
1.000001	-50000000

Thus, $\int_{-2}^{1} \frac{1}{(x-1)^3}\,dx$ is divergent. Since $\int_{-2}^{2} \frac{1}{(x-1)^3}\,dx = \int_{-2}^{1} \frac{1}{(x-1)^3}\,dx + \int_{1}^{2} \frac{1}{(x-1)^3}\,dx$,

we can conclude $\boxed{\int_{-2}^{2} \frac{1}{(x-1)^3}\,dx \text{ is divergent}}$ (there is no need to evaluate $\int_{1}^{2} \frac{1}{(x-1)^3}\,dx$).

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