Section 5.1/5.2: Areas and Distances – the Definite Integral

Practice HW from Stewart Textbook (not to hand in)
p. 352 # 3, 5, 9
p. 364 # 1, 3, 9-15 odd, 21-25 odd

Sigma Notation

The sum of *n* terms $a_1, a_2, \ldots, a_n$ is written as

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + \ldots + a_n$$, where *i* = the index of summation

Example 1: Find the sum $\sum_{i=1}^{4} 2i - 1$.

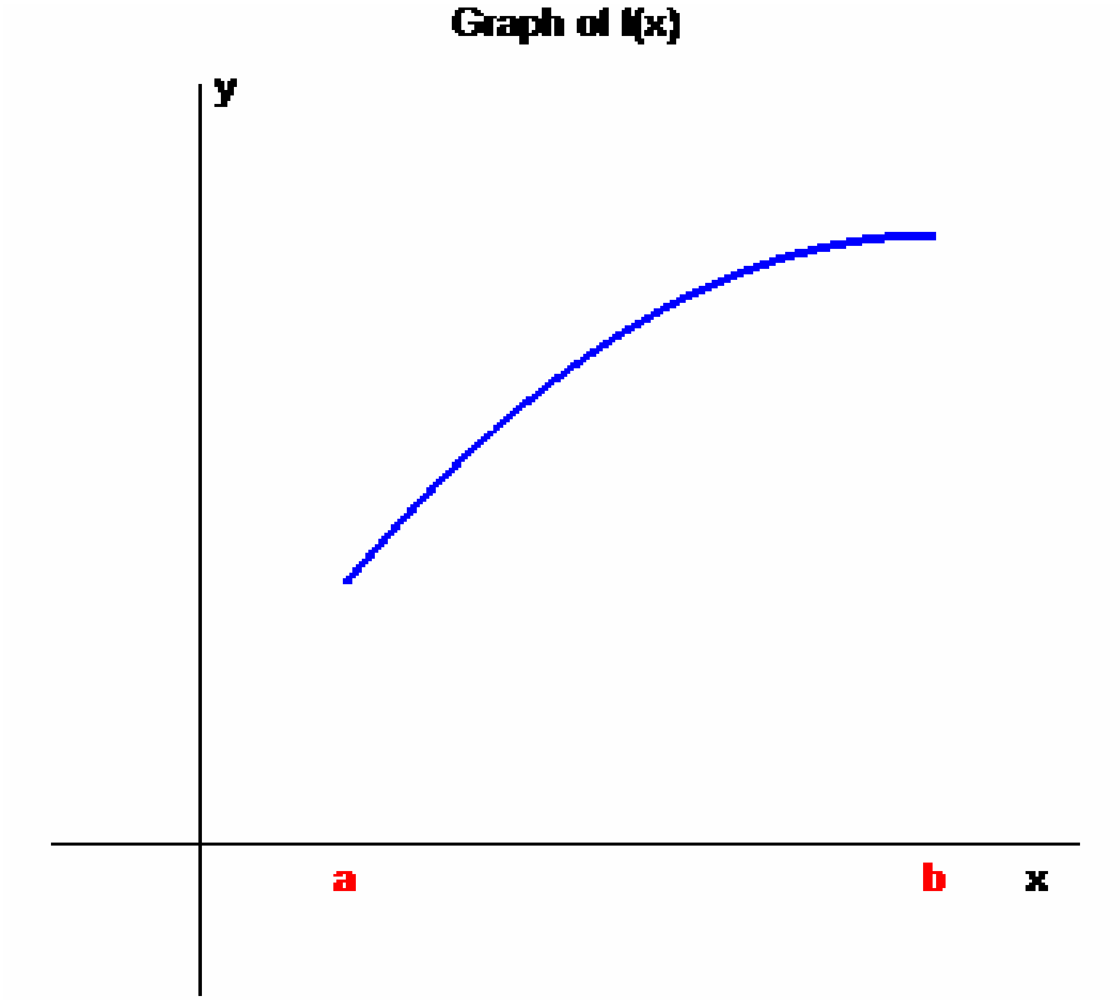
Solution:

∎

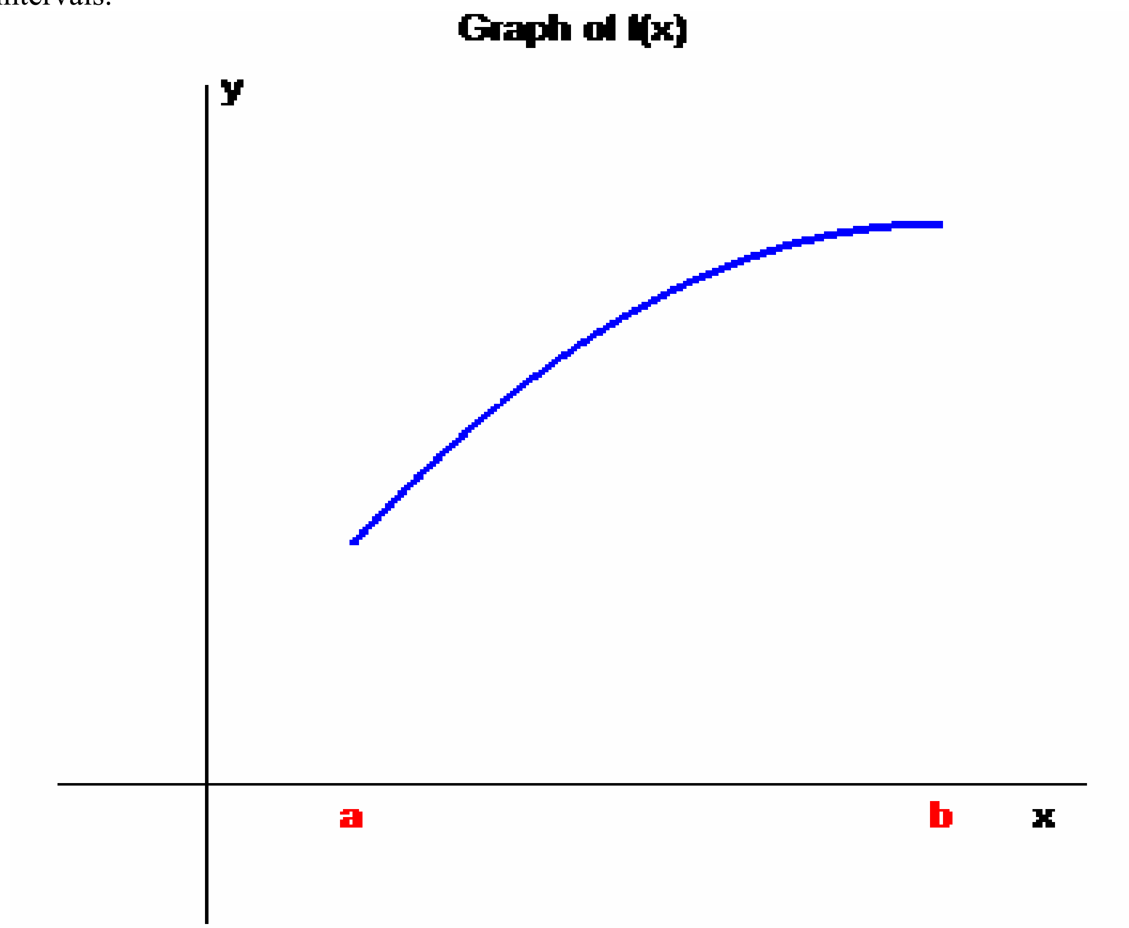
The Definite Integral

Suppose we have a function $f(x) \geq 0$ which is continuous, bounded, and increasing for $a \leq x \leq b$.

Goal: Suppose we desire to find the area A under the graph of f from $x = a$ to $x = b$.



To do this, we divide the interval for $a \le x \le b$ into n equal subintervals and form n rectangles (subintervals) under the graph of f. Let $x_0, x_1, x_2, \ldots x_n$ be the endpoints of each of the subintervals.



Here,

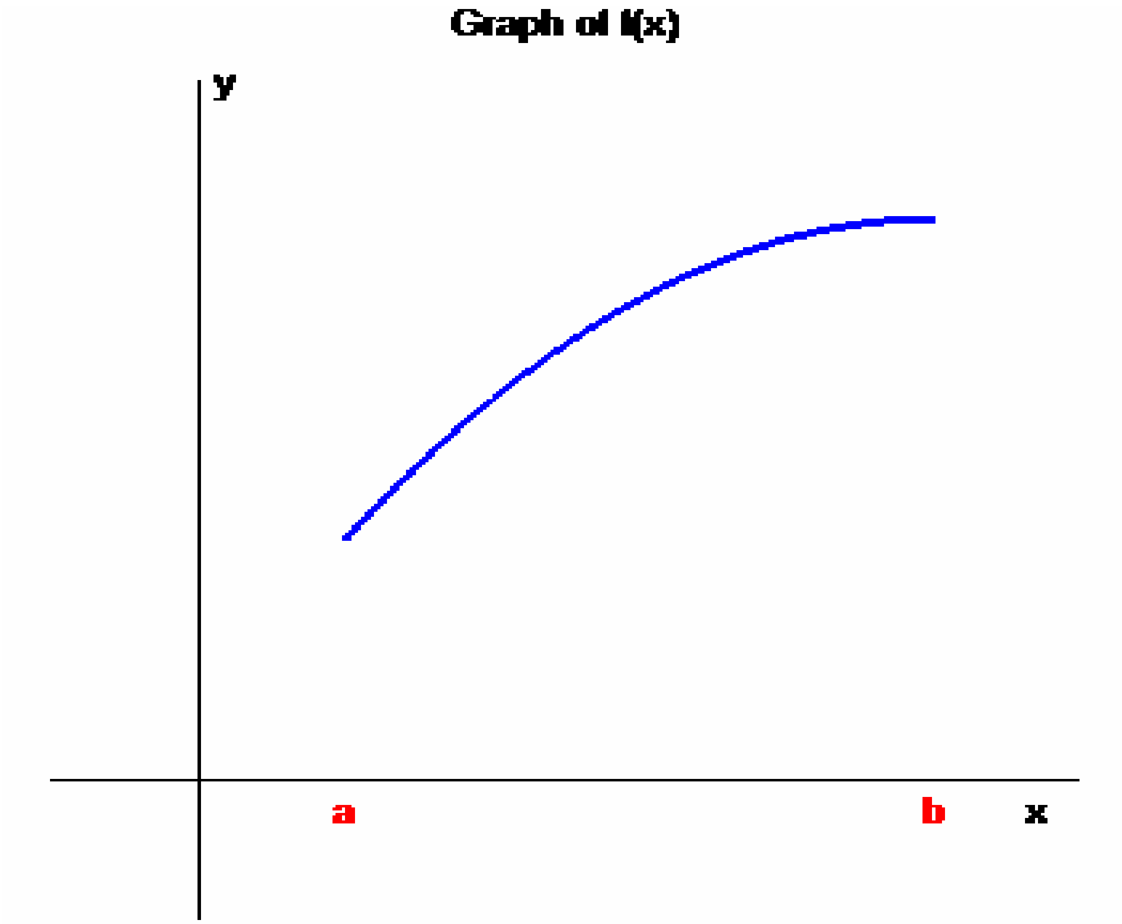
$$\begin{matrix}\text{Width of each} \\ \text{subinterval}\end{matrix} = \begin{matrix}\text{Width of each} \\ \text{rectangle}\end{matrix} = \Delta x = \frac{b-a}{n}$$

$$\text{Area of Rectangle 1} = (\text{Length})(\text{Width}) = f(x_0)\Delta x$$
$$\text{Area of Rectangle 2} = (\text{Length})(\text{Width}) = f(x_1)\Delta x$$
$$\vdots$$
$$\text{Area of Rectangle } n = (\text{Length})(\text{Width}) = f(x_{n-1})\Delta x$$

Summing up the area of the n rectangles, we see

$$\begin{matrix}\text{Total Area from} \\ a \le x \le b\end{matrix} \approx \begin{matrix}\text{Left Hand} \\ \text{endpoint sum}\end{matrix} = \sum_{i=0}^{n-1} f(x_i)\Delta x = f(x_0)\Delta x + f(x_1)\Delta x + \ldots + f(x_{n-1})\Delta x$$
$$= \left[f(x_0) + f(x_1) + \ldots + f(x_{n-1}) \right] \Delta x$$

We can also use the right endpoints of the intervals to find the length of the rectangles.



$$\text{Area of Rectangle } 1 = (\text{Length})(\text{Width}) = f(x_1)\Delta x$$
$$\text{Area of Rectangle } 2 = (\text{Length})(\text{Width}) = f(x_2)\Delta x$$
$$\vdots$$
$$\text{Area of Rectangle } n = (\text{Length})(\text{Width}) = f(x_n)\Delta x$$

Summing up the area of the n rectangles, we see

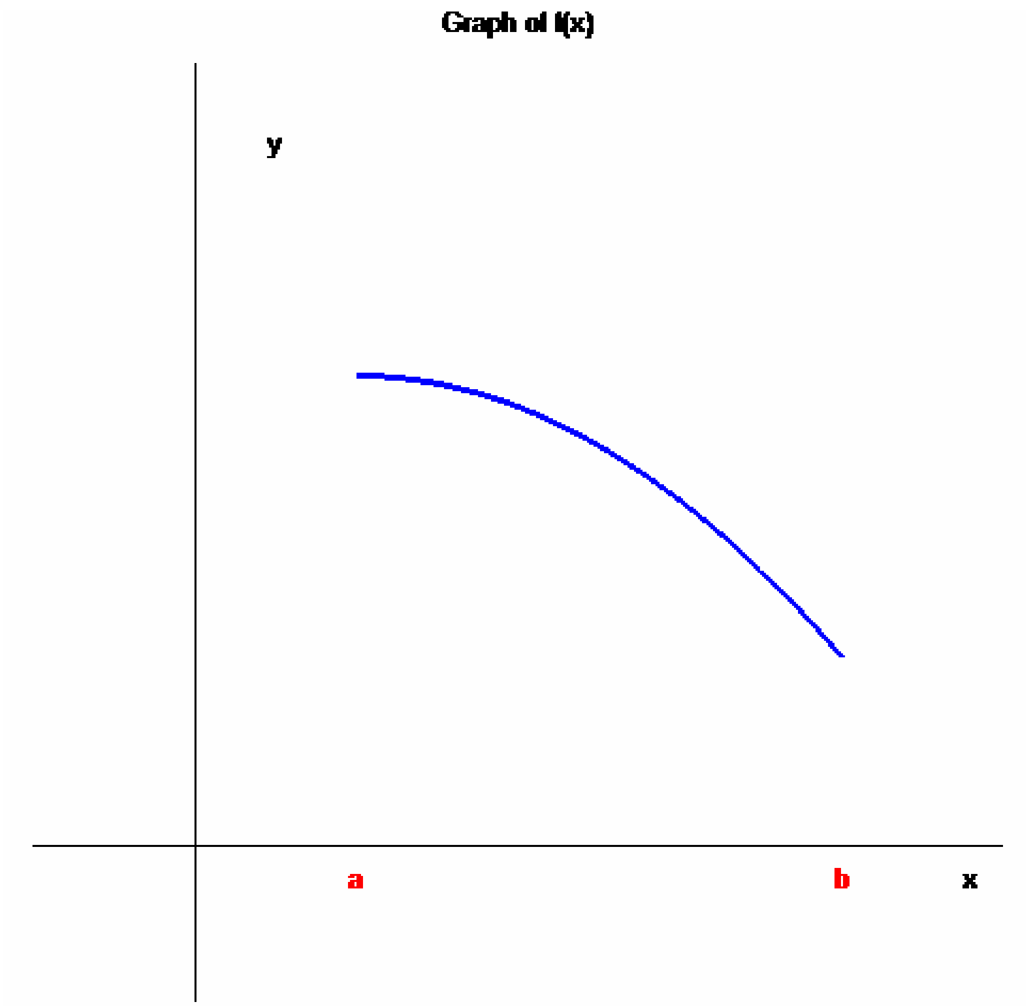
$$\begin{array}{c}\text{Total Area from} \\ a \le x \le b\end{array} \approx \begin{array}{c}\text{Right Hand} \\ \text{endpoint sum}\end{array} = \sum_{i=1}^{n} f(x_i)\Delta x = f(x_1)\Delta x + f(x_2)\Delta x + \ldots + f(x_n)\Delta x$$
$$= \left[f(x_1) + f(x_2) + \ldots + f(x_n) \right] \Delta x$$

Note: When f is increasing,

$$\text{Left Endpoint Sum} \le \begin{array}{c}\text{Total Area}\\ \text{from } a \le x \le b\end{array} \le \text{Right endpoint sum}$$

If f is decreasing,

$$\text{Right Endpoint Sum} \le \begin{array}{c}\text{Total Area}\\ \text{from } a \le x \le b\end{array} \le \text{Left endpoint sum}$$



$$\text{Right Endpoint Sum} \le \begin{array}{c}\text{Total Area}\\ \text{from } a \le x \le b\end{array} \le \text{Left endpoint sum}$$

In summary, if we divide the interval for $a \le x \le b$ into n equal subintervals and form n rectangles (subintervals) under the graph of f. Let $x_0, x_1, x_2, \ldots x_n$ be the endpoints of each of the subintervals.

$$\begin{matrix}\text{Total Area from} \\ a \le x \le b\end{matrix} \approx \begin{matrix}\text{Left Hand} \\ \text{endpoint sum}\end{matrix} = \sum_{i=0}^{n-1} f(x_i)\Delta x = f(x_0)\Delta x + f(x_1)\Delta x + \ldots + f(x_{n-1})\Delta x$$

$$= \left[f(x_0) + f(x_1) + \ldots + f(x_{n-1}) \right] \Delta x$$

$$\begin{matrix}\text{Total Area from} \\ a \le x \le b\end{matrix} \approx \begin{matrix}\text{Right Hand} \\ \text{endpoint sum}\end{matrix} = \sum_{i=1}^{n} f(x_i)\Delta x = f(x_1)\Delta x + f(x_2)\Delta x + \ldots + f(x_n)\Delta x$$

$$= \left[f(x_1) + f(x_2) + \ldots + f(x_n) \right] \Delta x$$

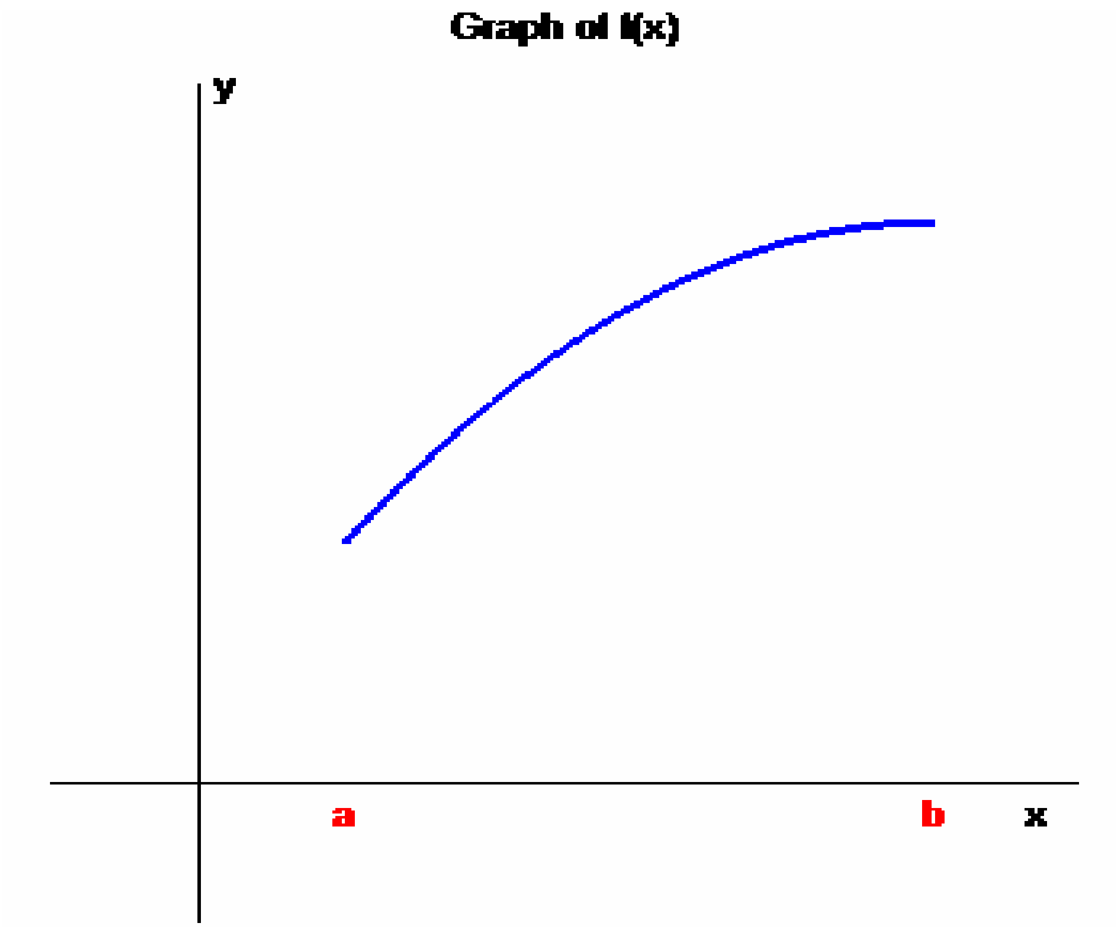
The endpoints of the n subintervals contained within $[a, b]$ are determined using the formula

$$x_i = x_0 + i\,\Delta x, \quad i = 1, 2, \ldots n \qquad \text{where} \quad x_0 = a \quad \text{and} \quad \Delta x = \frac{b-a}{n}$$

Example 2: Use the left and right endpoint sums to approximate the area under $y = x^2 + 1$ on the interval [0, 2] for $n = 4$ subintervals.

Solution:

Note: We can also approximate the area under a curve using the midpoint of the rectangles to find the rectangle's length.



$$\begin{matrix}\text{Total Area from} \\ a \le x \le b\end{matrix} \approx \begin{matrix}\text{Midpoint} \\ \text{sum}\end{matrix} = \sum_{i=1}^{n} f(\bar{x}_i)\Delta x = f(\bar{x}_1)\Delta x + f(\bar{x}_2)\Delta x + \ldots + f(\bar{x}_n)\Delta x$$

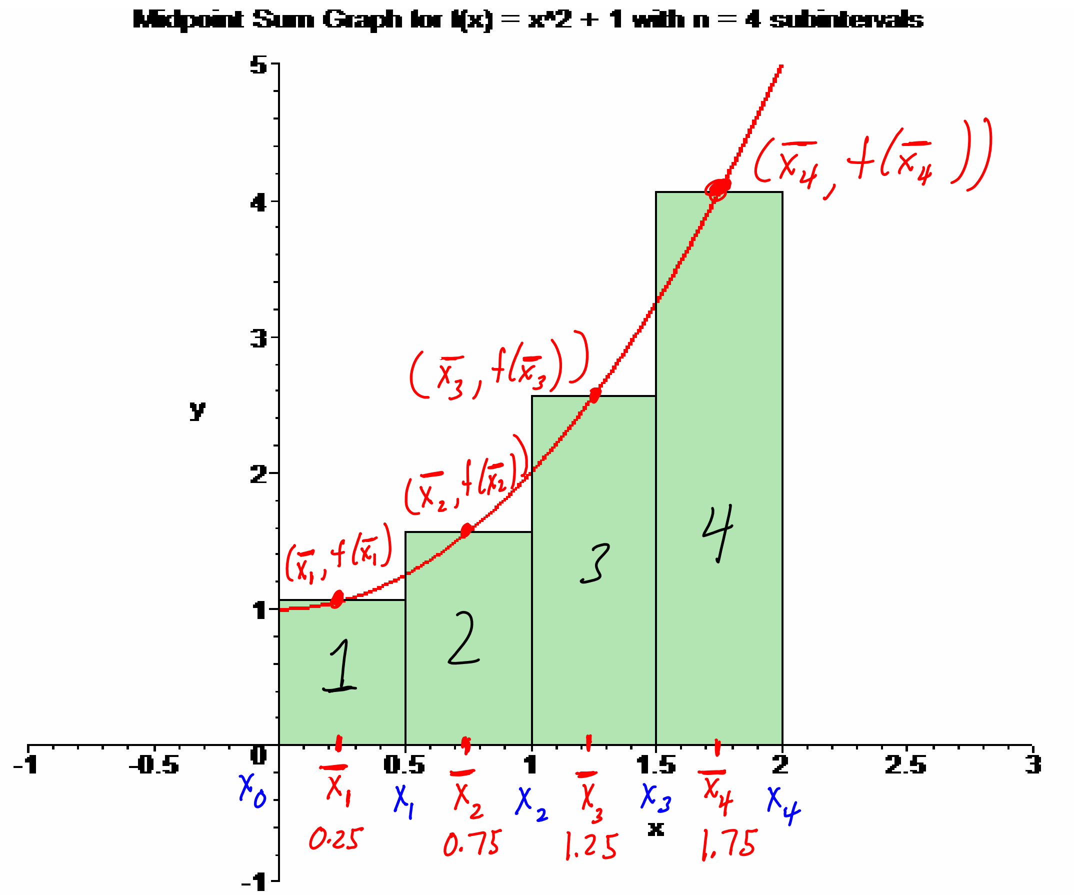
$$= \left[f(\bar{x}_1) + f(\bar{x}_2) + \ldots + f(\bar{x}_n) \right] \Delta x$$

where $x_i = x_0 + i\,\Delta x, \quad i = 1, 2, \ldots n$ where $\Delta x = \dfrac{b-a}{n}$ and

$$\bar{x}_i = \frac{1}{2}(x_{i-1} + x_i) = \text{midpoint of } [x_{i-1}, x_i],\ i = 1, 2, \ldots n$$

Example 3: Use the midpoint rule to approximate the area under $y = x^2 + 1$ on the interval [0, 2] for $n = 4$ subintervals.

Solution: Graphically, our goal is to find the area of the $n = 4$ rectangles for the interval $[a, b] = [0, 2]$ produced by the following graph.



In this problem,

$$\begin{matrix}\text{Width of a} \\ \text{Subinterval}\end{matrix} = \begin{matrix}\text{Width of a} \\ \text{Rectangle}\end{matrix} = \Delta x = \frac{b-a}{n} = \frac{2-0}{4} = \frac{1}{2} = 0.5 .$$

The endpoints of the $n = 4$ subintervals are calculated as follows using the formula $x_i = x_0 + i\Delta x$:

$$x_0 = a = 0$$
$$x_1 = x_0 + (1)\Delta x = 0 + (1)(0.5) = 0.5$$
$$x_2 = x_0 + (2)\Delta x = 0 + (2)(0.5) = 1$$
$$x_3 = x_0 + (3)\Delta x = 0 + (3)(0.5) = 1.5$$
$$x_4 = b = x_0 + (4)\Delta x = 0 + (4)(0.5) = 2$$

In this problem, we must find the midpoints of the $n = 4$ subintervals using the formula $\bar{x}_i = \frac{1}{2}\left(x_{i-1} + x_i\right)$. They are given as follows.

$$\bar{x}_1 = \frac{1}{2}(x_0 + x_1) = \frac{1}{2}(0 + 0.5) = \frac{1}{2}(0.5) = 0.25$$
$$\bar{x}_2 = \frac{1}{2}(x_1 + x_2) = \frac{1}{2}(0.5 + 1) = \frac{1}{2}(1.5) = 0.75$$
$$\bar{x}_3 = \frac{1}{2}(x_2 + x_3) = \frac{1}{2}(1 + 1.5) = \frac{1}{2}(2.5) = 1.25$$
$$\bar{x}_4 = \frac{1}{2}(x_3 + x_4) = \frac{1}{2}(1.5 + 2) = \frac{1}{2}(3.5) = 1.75$$

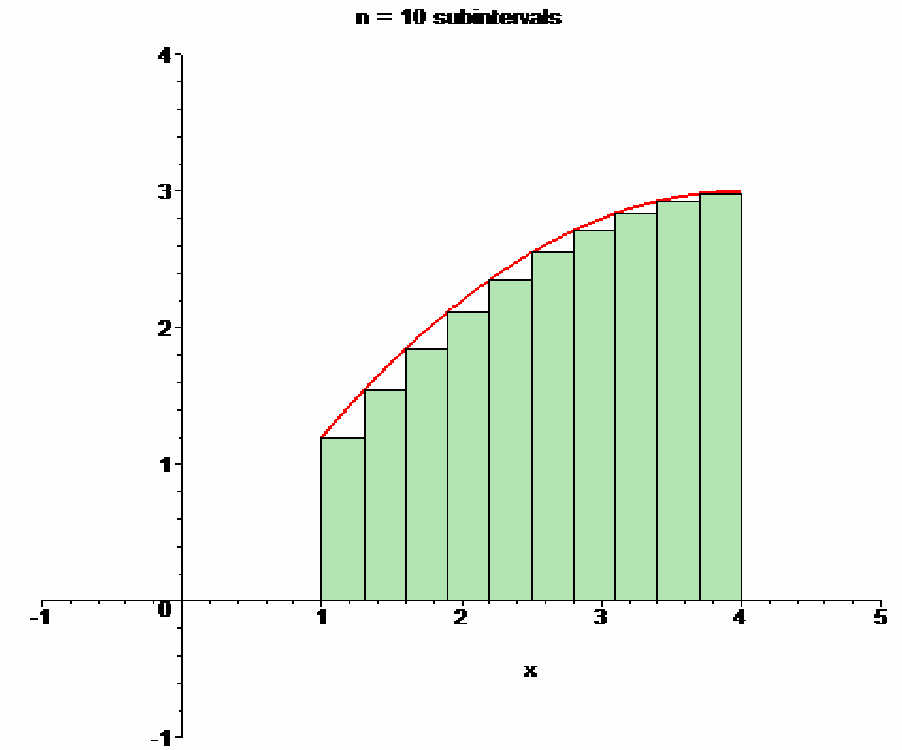
Hence, the heights of the rectangles using the function $f(x) = x^2 + 1$ are given as follows:

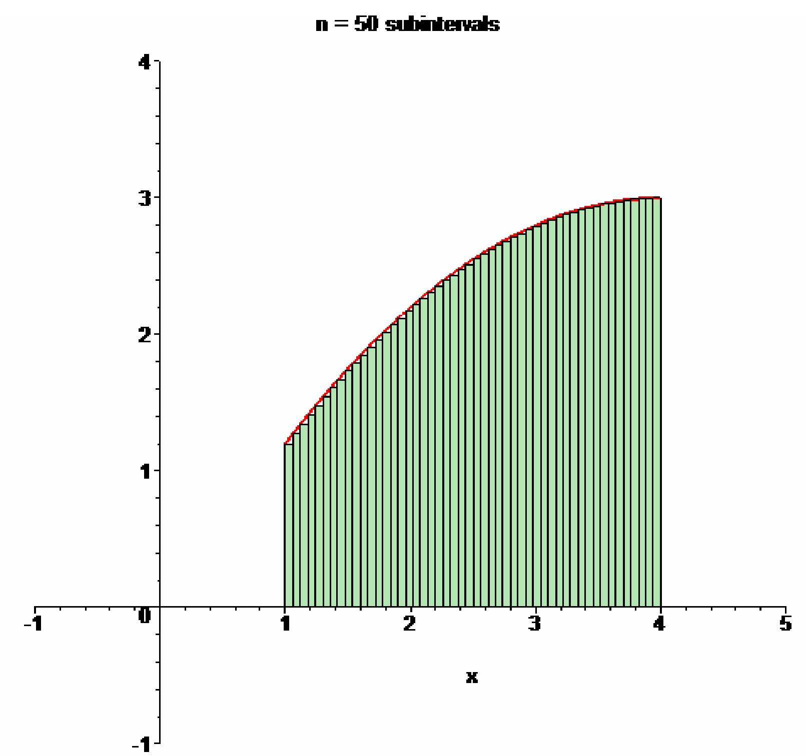
$$f(\bar{x}_1) = f(0.25) = (0.25)^2 + 1 = 0.0625 + 1 = 1.0625$$
$$f(\bar{x}_2) = f(0.75) = (0.75)^2 + 1 = 0.5625 + 1 = 1.5625$$
$$f(\bar{x}_3) = f(1.25) = (1.25)^2 + 1 = 1.5625 + 1 = 2.5625$$
$$f(\bar{x}_4) = f(1.75) = (1.75)^2 + 1 = 3.0625 + 1 = 4.0625$$

$$\begin{array}{c}\text{Midpoint Sum}\\ n = 4 \text{ subintervals}\end{array} = \begin{array}{c}\text{Area of}\\ \text{Rect 1}\end{array} + \begin{array}{c}\text{Area of}\\ \text{Rect 2}\end{array} + \begin{array}{c}\text{Area of}\\ \text{Rect 3}\end{array} + \begin{array}{c}\text{Area of}\\ \text{Rect 4}\end{array}$$
$$= \sum_{i=1}^{n} f(\bar{x}_i)\Delta x$$
$$= \sum_{i=1}^{4} f(\bar{x}_i)\Delta x$$
$$= \left[f(\bar{x}_1)\Delta x + f(\bar{x}_2)\Delta x + f(\bar{x}_3)\Delta x + f(\bar{x}_4)\Delta x \right]$$
$$= [1.0625 + 1.5625 + 2.5625 + 4.0625](0.5)$$
$$= (9.25)(0.5)$$
$$\boxed{= 4.625}$$

Note: The left endpoint, right endpoint, and midpoint sum rules are all special cases of what is known as Riemann sum.

Note: To increase accuracy, we need to increase the number of subintervals.





If we take n arbitrarily large, that is, take the limit of the left, midpoint, or right endpoint sums as $n \to \infty$, the left, midpoint, and right hand sums will be equal. The common value of the left, midpoint, and right endpoint sums is known as the definite integral.

Definition: The definite integral of f from a to b, written as

$$\int_a^b f(x)\,dx$$

is the limit of the left, midpoint, and right hand endpoint sums as $n \to \infty$. That is,

$$\underbrace{\lim_{n\to\infty}\sum_{i=0}^{n-1} f(x_i)\Delta x}_{\text{Left Hand Sum}} = \underbrace{\lim_{n\to\infty}\sum_{i=1}^{n} f(\bar{x}_i)\Delta x}_{\text{Midpoint Sum}} = \underbrace{\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x}_{\text{Right Hand Sum}} = \int_a^b f(x)\,dx$$

Notes

1. Each sum (left, midpoint, and right) is called a *Riemann* sum.
2. The endpoints a and b are called the limits of integration.
3. If $f(x) \geq 0$ and continuous on $[a, b]$, then

$$\int_a^b f(x)\,dx = \begin{matrix}\text{Area under } f \\ \text{from } a \leq x \leq b\end{matrix}$$

4. The endpoints of the n subintervals contained within $[a, b]$ are determined using the formula. Here, $\Delta x = \dfrac{b-a}{n}$.

   Left and right endpoints: $x_i = x_0 + i\,\Delta x, \quad i = 1, 2, \ldots n$

   Midpoints: $\bar{x}_i = \frac{1}{2}\left(x_{i-1} + x_i\right) = \text{midpoint of } [x_{i-1}, x_i],\ i = 1,2,\ldots n$

5. Evaluating a Riemann when the number of subintervals $n \to \infty$ requires some tedious algebra calculations. We will use Maple for this purpose. Taking a finite number of subintervals only approximates the definite integral.

Example 4: Use the left, right, and midpoint sums to approximate $\int_{-1}^{2} e^{-x^2}\,dx$ using $n = 5$ subintervals.

Solution: On this one, we begin by finding the subintervals and corresponding functional values for the endpoints of the $n = 5$ subintervals. First, note that the length of each subinterval for the interval [a, b] = [-1, 2] is

$$\Delta x = \frac{b-a}{n} = \frac{2-(-1)}{5} = \frac{3}{5} = 0.6$$

Hence, the endpoints of the $n = 5$ subintervals using the formula $x_i = x_0 + i\,\Delta x$ and the functional values using $f(x) = e^{-x^2}$ at these endpoints are:

$$x_0 = a = -1 \quad \Rightarrow \quad f(x_0) = f(-1) = e^{-(-1)^2} = e^{-1} \approx 0.3679\,.$$

$$x_1 = x_0 + (1)\Delta x = -1 + (1)(0.6) = -0.4 \quad \Rightarrow \quad f(x_1) = f(-0.4) = e^{-(-0.4)^2} = e^{-.16} \approx 0.8521$$

$$x_2 = x_0 + (2)\Delta x = -1 + (2)(0.6) = 0.2 \quad \Rightarrow \quad f(x_2) = f(0.2) = e^{-(0.2)^2} = e^{-0.04} \approx 0.9608$$

$$x_3 = x_0 + (3)\Delta x = -1 + (3)(0.6) = 0.8 \quad \Rightarrow \quad f(x_3) = f(0.8) = e^{-(0.8)^2} = e^{-0.64} \approx 0.5273$$

$$x_4 = x_0 + (4)\Delta x = -1 + (4)(0.6) = 1.4 \quad \Rightarrow \quad f(x_4) = f(1.4) = e^{-(1.4)^2} = e^{-1.96} \approx 0.1409$$

$$x_5 = b = x_0 + (5)\Delta x = -1 + (5)(0.6) = 2 \quad \Rightarrow \quad f(x_5) = f(2) = e^{-(2)^2} = e^{-4} \approx 0.0183$$

Then

$$\begin{aligned}
\text{Left Sum for } n = 5 \text{ subintervals} &= \sum_{i=0}^{n-1} f(x_i)\Delta x \\
&= \sum_{i=0}^{4} f(x_i)\Delta x \\
&= \left[f(x_0)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x \right] \\
&= \left[f(x_0) + f(x_1) + f(x_2) + f(x_3) + f(x_4) \right]\Delta x \\
&= [0.3679 + 0.8521 + 0.9608 + 0.5273 + 0.1409](0.6) \\
&= (2.849)(0.6) \\
&= \boxed{1.7094}
\end{aligned}$$

Right Sum for $n = 5$ subintervals $= \sum_{i=1}^{n} f(x_i)\Delta x$

$$= \sum_{i=1}^{5} f(x_i)\Delta x$$

$$= \left[f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x + f(x_5)\Delta x \right]$$

$$= \left[f(x_1) + f(x_2) + f(x_3) + f(x_4) + f(x_5) \right]\Delta x$$

$$= [0.8521 + 0.9608 + 0.5273 + 0.1409 + 0.0183](0.6)$$

$$= (2.4994)(0.6)$$

$$\boxed{= 1.49964}$$

To get the midpoint sum, we need to find the midpoints of the subintervals using the formula $\overline{x}_i = \frac{1}{2}(x_{i-1} + x_i)$. Recall from above that the endpoints of the subintervals are $x_0 = -1$, $x_1 = -0.4$, $x_2 = 0.2$, $x_3 = 0.8$, $x_4 = 1.4$, and $x_5 = 2$. The following calculation finds the midpoints and evaluates the functional values at these midpoints.

$$\overline{x}_1 = \frac{1}{2}(x_0 + x_1) = \frac{1}{2}(-1 + -0.4) = -0.7 \quad \Rightarrow \quad f(\overline{x}_1) = f(-0.7) = e^{-(-0.7)^2} = e^{-0.49} \approx 0.6126$$

$$\overline{x}_2 = \frac{1}{2}(x_1 + x_2) = \frac{1}{2}(-0.4 + 0.2) = \frac{1}{2}(-0.2) = -0.1 \quad \Rightarrow \quad f(\overline{x}_2) = f(-0.1) = e^{-(-0.1)^2} = e^{-0.01} \approx 0.99$$

$$\overline{x}_3 = \frac{1}{2}(x_2 + x_3) = \frac{1}{2}(0.2 + 0.8) = \frac{1}{2}(1) = 0.5 \quad \Rightarrow \quad f(\overline{x}_3) = f(0.5) = e^{-(0.5)^2} = e^{-0.25} \approx 0.7788$$

$$\overline{x}_4 = \frac{1}{2}(x_3 + x_4) = \frac{1}{2}(0.8 + 1.4) = \frac{1}{2}(2.2) = 1.1 \quad \Rightarrow \quad f(\overline{x}_4) = f(1.1) = e^{-(1.1)^2} = e^{-1.21} \approx 0.2982$$

$$\overline{x}_5 = \frac{1}{2}(x_4 + x_5) = \frac{1}{2}(1.4 + 2) = \frac{1}{2}(3.4) = 1.7 \quad \Rightarrow \quad f(\overline{x}_5) = f(1.7) = e^{-(1.7)^2} = e^{-2.89} \approx 0.0556$$

Midpoint Sum for $n = 5$ subintervals $= \sum_{i=1}^{n} f(\overline{x}_i)\Delta x$

$$= \sum_{i=1}^{5} f(\overline{x}_i)\Delta x$$

$$= \left[f(\overline{x}_1)\Delta x + f(\overline{x}_2)\Delta x + f(\overline{x}_3)\Delta x + f(\overline{x}_4)\Delta x + f(\overline{x}_5)\Delta x \right]$$

$$= \left[f(\overline{x}_1) + f(\overline{x}_2) + f(\overline{x}_3) + f(\overline{x}_4) + f(\overline{x}_5) \right]\Delta x$$

$$= [0.6126 + 0.99 + 0.7788 + 0.2982 + 0.0556](0.6)$$

$$= (2.73516)(0.6)$$

$$\boxed{= 1.641096}$$

To increase accuracy, we need to make the number of subintervals n (the number of rectangles) larger. Maple can be used to do this. If we let $n \to \infty$, then the resulting limit of the left endpoint, midpoint, or right endpoint sum will give the exact value of the definite integral. Recall that

$$\underbrace{\lim_{n\to\infty} \sum_{i=0}^{n-1} f(x_i)\Delta x}_{\text{Left Hand Sum}} = \underbrace{\lim_{n\to\infty} \sum_{i=1}^{n} f(\bar{x}_i)\Delta x}_{\text{Midpoint Sum}} = \underbrace{\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x}_{\text{Right Hand Sum}} = \int_a^b f(x)\,dx$$

The following example will illustrate how this infinite limit can be set up using the right hand sum.

Example 3: Set up the right hand sum limit for finding the exact value of $\int_0^2 (x^2 + 1)\,dx$ for n total subintervals.

Solution:

∎

Summarizing, using Maple, we can find the following information for approximating $\int_0^2 (x^2+1)\,dx$ and $\int_{-1}^2 e^{-x^2}\,dx$.

$\int_0^2 (x^2+1)\,dx$	**Left Endpoint**	**Midpoint**	**Right Endpoint**
***n*=4 subintervals**	**3.75**	**4.625**	**5.75**
***n*=10 subintervals**	**4.28**	**4.66**	**5.08**
***n*=30 subintervals**	**4.534814815**	**4.665925926**	**4.801481482**
***n*=100 subintervals**	**4.626800000**	**4.666600000**	**4.706800000**
***n*=1000 subintervals**	**4.662668000**	**4.666666000**	**4.670668000**
Exact Value $n \to \infty$	$\frac{14}{3} \approx 4.\overline{6}$	$\frac{14}{3} \approx 4.\overline{6}$	$\frac{14}{3} \approx 4.\overline{6}$

$\int_{-1}^2 e^{-x^2}\,dx$	**Left Endpoint**	**Midpoint**	**Right Endpoint**
***n*=5 subintervals**	**1.709378108**	**1.641150303**	**1.499639827**
***n*=10 subintervals**	**1.675264205**	**1.631946545**	**1.570395065**
***n*=30 subintervals**	**1.645709427**	**1.629242706**	**1.610753045**
***n*=100 subintervals**	**1.634088303**	**1.628935862**	**1.623601388**
***n*=1000 subintervals**	**1.629429262**	**1.628905837**	**1.628380572**
Value $n \to \infty$	**1.628905524**	**1.628905524**	**1.628905524**