Section 5.7: Additional Techniques of Integration

Practice HW from Stewart Textbook (not to hand in)
p. 404 # 1-5 odd, 9-27 odd

Integrals Involving Powers of Sine and Cosine

Two Types

1. Odd Powers of Sine and Cosine: Attempt to write the sine or cosine term with the lowest odd power in terms of an odd power times the square of sine and cosine. Then rewrite the squared term using the Pythagorean identity $\sin^2 x + \cos^2 x = 1$.

Example 1: Integrate $\int \cos^5 x \, \sin^3 x \, dx$

Solution:

∎

2. Only Even Powers of Sine and Cosine: Use the identity $\sin^2 u = \frac{1-\cos 2u}{2} = \frac{1}{2}(1-\cos 2u)$ and $\cos^2 u = \frac{1+\cos 2u}{2} = \frac{1}{2}(1+\cos 2u)$. Note in these formulas the initial angle is always doubled.

Example 2: Integrate $\int_0^{\frac{\pi}{2}} \sin^2 3x \; dx$

Solution: Since the sine term is even, we use the identities $\sin^2 u = \frac{1}{2}(1-\cos 2u)$ to rewrite the integral. Hence we have

$$\int_0^{\frac{\pi}{2}} \sin^2 3x \; dx = \int_0^{\frac{\pi}{2}} \frac{1}{2}(1-\cos 6x)\, dx \qquad \text{Use fact that } \sin^2 3x = \frac{1}{2}(1-\cos 6x)$$

$$= \int_0^{\frac{\pi}{2}} (\frac{1}{2} - \frac{1}{2}\cos 6x)\, dx \qquad \text{(Distribute the } \frac{1}{2}\text{)}$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{2}\, dx - \int_0^{\frac{\pi}{2}} \frac{1}{2}\cos 6x\, dx \qquad \text{(Break into separate integrals)}$$

$$= \left(\frac{1}{2}x - \frac{1}{2}(\frac{1}{6})\sin 6x\right)\Big|_0^{\frac{\pi}{2}} \qquad \text{(By } u-du \text{ substitution, } \int \cos 6x\, dx = \frac{1}{6}\sin 6x + C\text{)}$$

$$= \left(\frac{1}{2}x - \frac{1}{12}\sin 6x\right)\Big|_0^{\frac{\pi}{2}}$$

$$= \left(\frac{1}{2}(\frac{\pi}{2}) - \frac{1}{12}\sin 6(\frac{\pi}{2})\right) - \left(\frac{1}{2}(0) - \frac{1}{12}\sin 6(0)\right)$$

$$= \left(\frac{\pi}{4} - \frac{1}{12}\sin 3\pi\right) - \left(0 - \frac{1}{12}\sin 0\right)$$

$$= \left(\frac{\pi}{4} - \frac{1}{12}(0)\right) - \left(0 - \frac{1}{12}(0)\right) \qquad \text{(Note that } \sin 3\pi = 0 \text{ and } \sin 0 = 0\text{)}$$

$$= \frac{\pi}{4} + 0 - 0$$

$$\boxed{= \frac{\pi}{4}}$$

■

Trigonometric Substitution

Good for integrating functions with complicated radical expressions.

Useful Identities

1. $\sin^2\theta + \cos^2\theta = 1 \quad \Rightarrow \quad \cos^2\theta = 1 - \sin^2\theta$
2. $\tan^2\theta + 1 = \sec^2\theta \quad \Rightarrow \quad \sec^2\theta = 1 + \tan^2\theta$ and $\tan^2\theta = \sec^2\theta - 1$.

<u>Trigonometric Substitution Forms</u>

Let *x* be a variable quantity and *a* a real number.

1. For integrals involving $\sqrt{a^2 - x^2}$, let $x = a\sin\theta$.
2. For integrals involving $\sqrt{a^2 + x^2}$, let $x = a\tan\theta$.
3. For integrals involving $\sqrt{x^2 - a^2}$, let $x = a\sec\theta$.

Example 3: Integrate $\int \frac{1}{x^2\sqrt{25-x^2}}\,dx$

Solution:

[redacted]

Partial Fractions

Decomposes a rational function into simpler rational functions that are easier to integrate. Essentially undoes the process of finding a common denominator of fractions.

Partial Fractions Process

1. Check to make sure the degree of the numerator is less than the degree of the denominator. If not, need to divide by long division.
2. Factor the denominator into linear or quadratic factors of the form

   Linear: $(px+q)^m$ Quadratic: $(ax^2+bx+c)^m$

3. For linear functions:

$$\frac{f(x)}{(px+q)^m} = \frac{A_1}{px+q} + \frac{A_2}{(px+q)^2} + \frac{A_3}{(px+q)^3} + \ldots + \frac{A_{m-1}}{(px+q)^{m-1}} + \frac{A_m}{(px+q)^m}$$ where

   $A_1, A_2, A_3, \ldots, A_m$ are real numbers.

4. For Quadratic Factors:

$$\frac{g(x)}{(ax^2+bx+c)^n} = \frac{B_1x+C_1}{ax^2+bx+c} + \frac{B_2x+C_2}{(ax^2+bx+c)^2} + \frac{B_3x+C_3}{(ax^2+bx+c)^3} + \ldots + \frac{B_nx+C_n}{(ax^2+bx+c)^n}$$

where $B_1, B_2, B_3, \ldots, B_n$ and $C_1, C_2, C_3, \ldots, C_n$.

Integrating Functions With Linear Factors Using Partial Fractions

1. Substitute the roots of the distinct linear factors of the denominator into the basic equation (the equation obtained after eliminating the fractions on both sides of the equation) and find the resulting constants.
2. For repeated linear factors, use the coefficients found in step 1 and substitute other convenient values of *x* to find the other coefficients.
3. Integrate each term.

Example 4: Integrate $\int \frac{x-2}{x^2+4x+3}\,dx$

Solution:

Integrating terms using Partial Fractions with Irreducible Quadratic Terms $ax^2 + bx + c$ (quadratic terms that cannot be factored) in the Denominator.

1. Expand the basic equation and combine the like terms of *x*.
2. Equate the coefficients of like powers and solve the resulting system of equations.
3. Integrate.

Useful Derivation of Inverse Tangent Integration Formula:

$$\int \frac{1}{x^2 + a^2}\,dx = \int \frac{1}{a^2\left[\left(\frac{x}{a}\right)^2 + 1\right]}\,dx$$

$u - du$ Substitution

Let $u = \frac{x}{a} = \frac{1}{a}x$

$du = \frac{1}{a}dx$

$a\,du = dx$

$$= \frac{1}{a^2}\int \frac{1}{\left[\left(\frac{x}{a}\right)^2 + 1\right]}\,dx$$

$$= \frac{1}{a^2}\int \frac{1}{u^2 + 1}\,a\,du$$

$$= \frac{a}{a^2}\int \frac{1}{u^2 + 1}\,du$$

$$= \frac{1}{a}\arctan u + C = \frac{1}{a}\arctan\frac{x}{a} + C$$

Generalized Inverse Tangent Integration Formula

$$\int \frac{1}{x^2 + a^2}\,dx = \frac{1}{a}\arctan\frac{x}{a} + C$$

Example 5: Integrate $\int \frac{8x}{(x-1)(x^2+4)}\,dx$

Solution:

[redacted]

Example 6: Find the partial fraction expansion of

$$\frac{2x}{x^3(x-3)^4(x^2+1)(x^2+4)^3}$$

Solution:

∎

Fact: If the degree of the numerator (highest power of *x*) is bigger than or equal to the degree of the denominator, must use long polynomial division to simplify before integrating the function.

Example 7: Integrate $\int \frac{x^2}{x+4}\,dx$

Solution:

■

Integration Technique Chart