Section 5.9 Approximate Integration

Practice HW from Stewart Textbook (not to hand in)
p. 421 # 3 – 15 odd

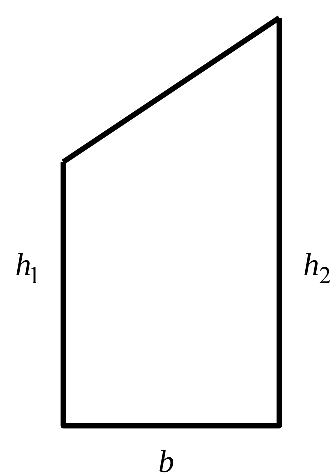
Note: Many functions cannot be integrated using the basic integration formulas or with any technique of integration (substitution, parts, etc.).

Examples: $f(x) = e^{-x^2}$, $f(x) = \sin x^2$.

As a result, we cannot use the Fundamental Theorem of Calculus to determine the area under the curve. We must use numerical techniques. We have already seen how to do this using left, right, midpoint sums. In this section, we will examine two other techniques, which in general will produce more accuracy with less work, to approximate definite integrals.

Trapezoid Rule

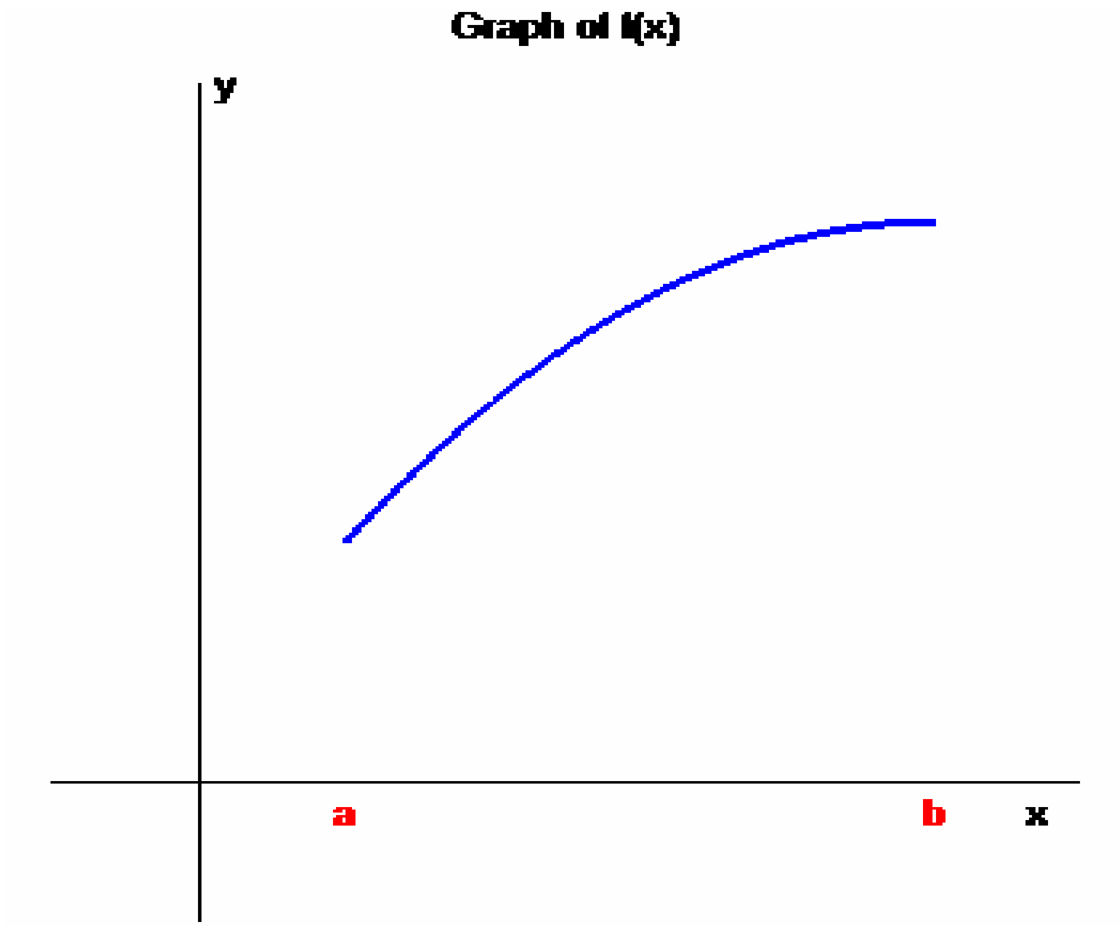
The idea behind the trapezoid rule is to approximate the area under a curve using the area of trapezoids. Suppose we have the following diagram of a trapezoid.



Recall that the area of the trapezoid is given by the following formula.

$$\text{Area of trapezoid} = \text{(base)(Average of the height)} = b\left(\frac{h_1 + h_2}{2}\right)$$

Suppose we have a function $f(x) \geq 0$ which is continuous and bounded for $a \leq x \leq b$. Suppose we desire to find the area A under the graph of f from $x = a$ to $x = b$. To do this, we divide the interval for $a \leq x \leq b$ into n equal subintervals of width $\Delta x = \frac{b-a}{n}$ and form n trapezoids (subintervals) under the graph of f. Let $a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b$ be the endpoints of each of the subintervals.



Here,

$$\text{Area of Trapezoid } 1 = \Delta x\left(\frac{f(x_0)+f(x_1)}{2}\right) = \frac{b-a}{n}\left(\frac{f(x_0)+f(x_1)}{2}\right)$$

$$\text{Area of Trapezoid } 2 = \Delta x\left(\frac{f(x_1)+f(x_2)}{2}\right) = \frac{b-a}{n}\left(\frac{f(x_1)+f(x_2)}{2}\right)$$

$$\vdots$$

$$\text{Area of Trapezoid n} = \Delta x\left(\frac{f(x_{n-1})+f(x_n)}{2}\right) = \frac{b-a}{n}\left(\frac{f(x_{n-1})+f(x_n)}{2}\right)$$

Summing up the area of the n trapezoids, we see

$$\begin{array}{c}\text{Total Area}\\ \text{under } f\\ [a,b]\end{array} \approx \begin{array}{c}\text{Trapezoid}\\ \text{Area sum}\end{array}$$

$$= \frac{b-a}{n}\left[\frac{f(x_0)+f(x_1)}{2}+\frac{f(x_1)+f(x_2)}{2}+\frac{f(x_2)+f(x_3)}{2}+\ldots+\frac{f(x_{n-1})+f(x_n)}{2}\right]$$

$$= \frac{b-a}{2n}\left[f(x_0)+2f(x_1)+2f(x_2)\ldots+2f(x_{n-1})+f(x_n)\right]$$

Trapezoid Rule

The definite integral of a continuous function f on the interval $[a, b]$ can be approximated using n subintervals as follows:

$$\int_a^b f(x)\,dx \approx T_n = \frac{b-a}{2n}\left[f(x_0)+2f(x_1)+2f(x_2)\ldots+2f(x_{n-1})+f(x_n)\right]$$

where $\Delta x = \dfrac{b-a}{n}$, $x_0 = a$, and $x_i = x_0 + i\Delta x$.

Example 1: Use the trapezoid rule to approximate $\int_0^2 (x^2 + 1)\, dx$.

Solution:

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Simpson's Rule

Use's a sequence of quadratic functions (parabolas) to approximate the definite integral.
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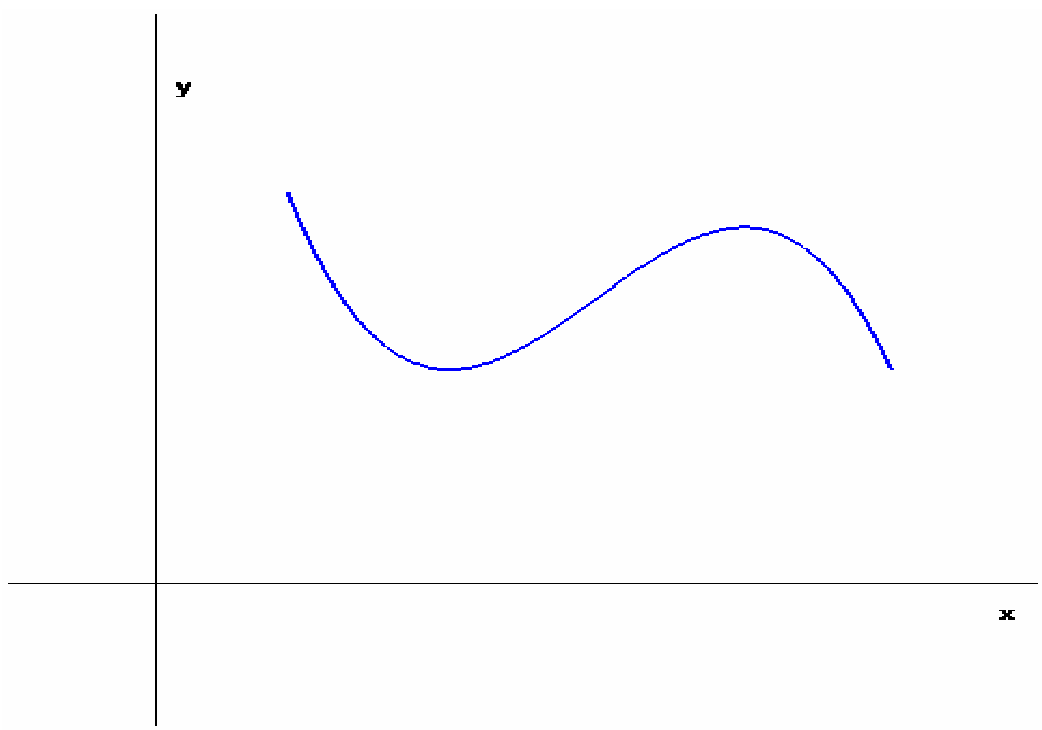
Theorem: Given a quadratic function $p(x) = Ax^2 + Bx + C$, then

$$\int_a^b p(x)\,dx = \frac{b-a}{6}\left[p(a) + 4p\left(\frac{a+b}{2}\right) + p(b) \right].$$

We again partition the interval [*a*, *b*] into *n* equal subintervals of length $\Delta x = \frac{b-a}{n}$. Note that *n* must be even. Here, we have

$$a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b, \qquad n \text{ is even.}$$



On each double subinterval $[x_{i-2}, x_i]$, we approximate the area under f by approximating the area under the polynomial $p(x)$.

$$\text{Area from } [x_0, x_2] = \int_{x_0}^{x_2} f(x)\,dx \approx \int_{x_0}^{x_2} p(x)\,dx$$

$$= \frac{x_2 - x_0}{6}\left[p(x_0) + 4p(\frac{x_0 + x_2}{2}) + p(x_2) \right]$$

$$= \frac{2\Delta x}{6}\left[p(x_0) + 4p(x_1) + p(x_2)\right]$$

$$= \frac{b-a}{3n}[\left[f(x_0) + 4f(x_1) + f(x_2)\right]$$

Similarly,

$$\text{Area from } [x_2, x_4] \approx \frac{b-a}{3n}[\left[f(x_2) + 4f(x_3) + f(x_4)\right],$$

$$\text{Area from } [x_4, x_6] \approx \frac{b-a}{3n}[\left[f(x_4) + 4f(x_5) + f(x_6)\right], \text{ etc.}$$

Repeating this process for all subintervals, we get the following rule.

Simpson's Rule

Let f be continuous on $[a, b]$. For an even number of subintervals,

$$\int_a^b f(x)\,dx \approx \frac{b-a}{3n}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \ldots + 4f(x_{n-1}) + f(x_n)\right]$$

where $\Delta x = \dfrac{b-a}{n}$, $x_0 = a$, and $x_i = x_0 + i\Delta x$.

Example 2: Use Simpson's rule to approximate $\int_0^2 (x^2 + 1)\, dx$.

Solution:

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Example 3: Use the trapezoidal and Simpson's rule to approximate $\int_{-1}^{1} \sin x^2 \, dx$ using n = 8 subintervals.

Solution: On this problem, we begin by finding the subintervals and corresponding functional values for the endpoints of the n = 8 subintervals. First, note that the length of each subinterval for the interval [a, b] = [-1, 1] is

$$\Delta x = \frac{b-a}{n} = \frac{1-(-1)}{8} = \frac{2}{8} = 0.25$$

Henc2e, the endpoints of the n = 8 subintervals using the formula $x_i = x_0 + i \, \Delta x$ and the functional values using $f(x) = \sin x^2$ at these endpoints are:

$x_0 = a = -1 \quad \Rightarrow \quad f(x_0) = f(-1) = \sin(-1)^2 = \sin(1) \approx 0.8415$.

$x_1 = x_0 + (1)\Delta x = -1 + (1)(0.25) = -0.75 \quad \Rightarrow \quad f(x_1) = f(-0.75) = \sin((-0.75)^2) \approx 0.5333$

$x_2 = x_0 + (2)\Delta x = -1 + (2)(0.25) = -0.5 \quad \Rightarrow \quad f(x_2) = f(-0.5) = \sin((-0.5)^2) \approx 0.2474$

$x_3 = x_0 + (3)\Delta x = -1 + (3)(0.25) = -0.25 \quad \Rightarrow \quad f(x_3) = f(-0.25) = \sin((-0.25)^2) \approx 0.0625$

$x_4 = x_0 + (4)\Delta x = -1 + (4)(0.25) = 0 \quad \Rightarrow \quad f(x_4) = f(0) = \sin((0)^2) = 0$

$x_5 = x_0 + (5)\Delta x = -1 + (5)(0.25) = 0.25 \quad \Rightarrow \quad f(x_5) = f(0.25) = \sin((0.25)^2) \approx 0.0625$

$x_6 = x_0 + (6)\Delta x = -1 + (6)(0.25) = 0.5 \quad \Rightarrow \quad f(x_6) = f(0.5) = \sin((0.5)^2) \approx 0.2474$

$x_7 = x_0 + (1)\Delta x = -1 + (7)(0.25) = 0.75 \quad \Rightarrow \quad f(x_1) = f(0.75) = \sin((0.75)^2) \approx 0.5333$

$x_8 = b = x_0 + (8)\Delta x = -1 + (8)(0.25) = 1 \quad \Rightarrow \quad f(x_8) = f(1) = \sin((1)^2) \approx 0.8415$

Then the <u>trapezoidal</u> rule says that

$$\begin{aligned}
\int_{-1}^{1} \sin x^2 \, dx &\approx \frac{b-a}{2n}\left[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + 2f(x_4) + 2f(x_5) + 2f(x_6) + 2f(x_7) + f(x_8)\right] \\
&= \frac{1-(-1)}{2(8)}\left[0.8415 + 2(0.5333) + 2(0.2474) + 2(0.0625) + 2(0) + 2(0.0625) + 2(0.2474) + 2(0.5333) + 0.8415\right] \\
&= \frac{2}{16}(5.0558) = \frac{1}{8}(5.0558) = \boxed{0.6320}
\end{aligned}$$

(continued on next page)

<u>Simpson's</u> rule says that

$$\int_{-1}^{1} \sin x^2 \, dx \approx \frac{b-a}{3n}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + 4f(x_5) + 2f(x_6) + 4f(x_7) + f(x_8) \right]$$

$$= \frac{1-(-1)}{3(8)}\left[0.8415 + 4(0.5333) + 2(0.2474) + 4(0.0625) + 2(0) + 4(0.0625) + 2(0.2474) + 4(0.5333) + 0.8415 \right]$$

$$= \frac{2}{24}(7.439) = \frac{1}{12}(7.439) \boxed{= 0.6199}$$

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