Section 6.2: Volumes

Practice HW from Stewart Textbook (not to hand in)
p. 457 # 1-13 odd, 49

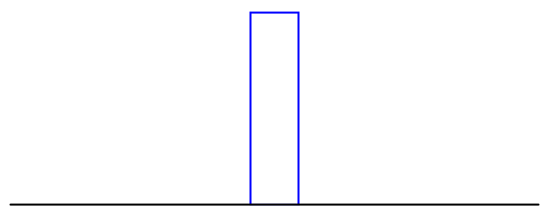
Solids of Revolution

In this section, we want to examine how to find the volume of a solid of revolution, which is formed by rotating a region in a plane about a line.

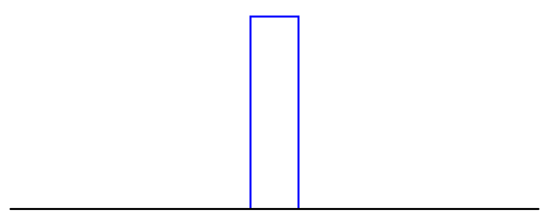
Examples: Figures 5, 6 p. 450

Disc Method

Consider a rectangle of length *R* and width *w*.

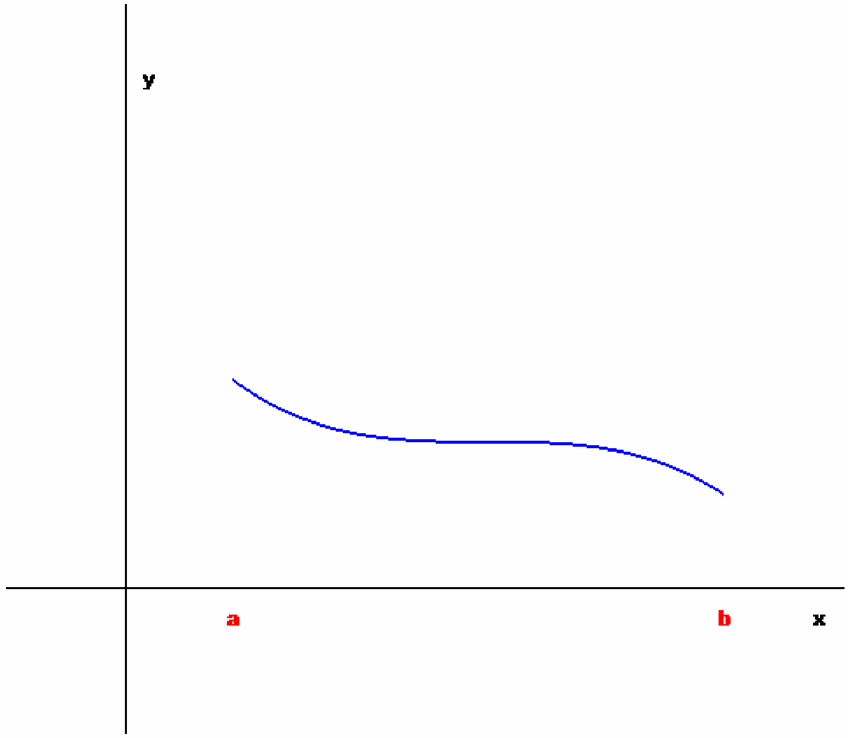


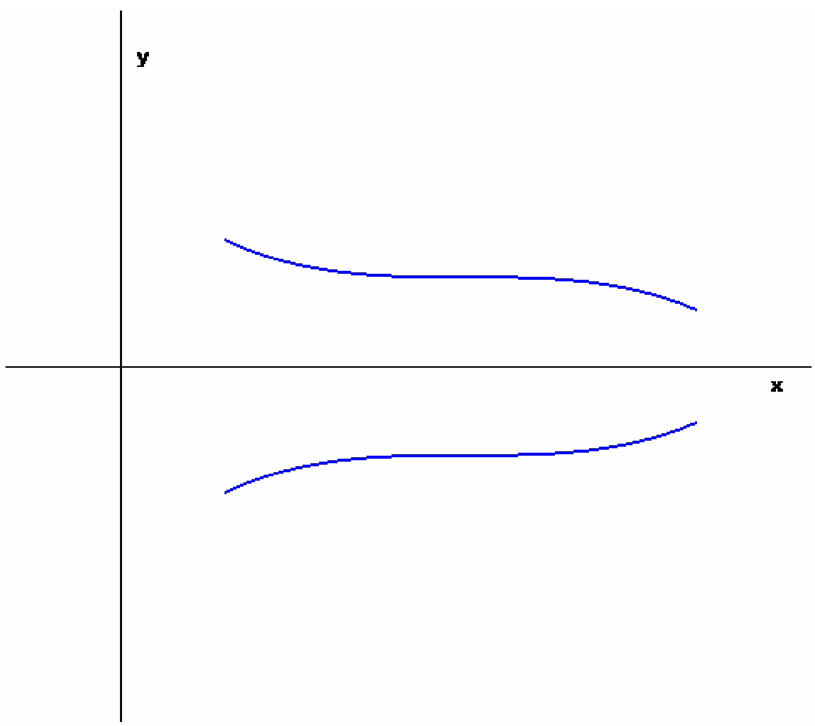
If we rotate this rectangle about this axis, we sweep out a circular disk.



Volume of Disk = (Area of Disk)(Width of Disk) = $\pi R^2 w$.

Now, suppose we have a function $R(x)$ and rotate it about the x-axis.





$$\Delta V = \text{Volume of Disk } i = (\pi R^2)\Delta x = \pi[R(x_i)]^2 \Delta x$$

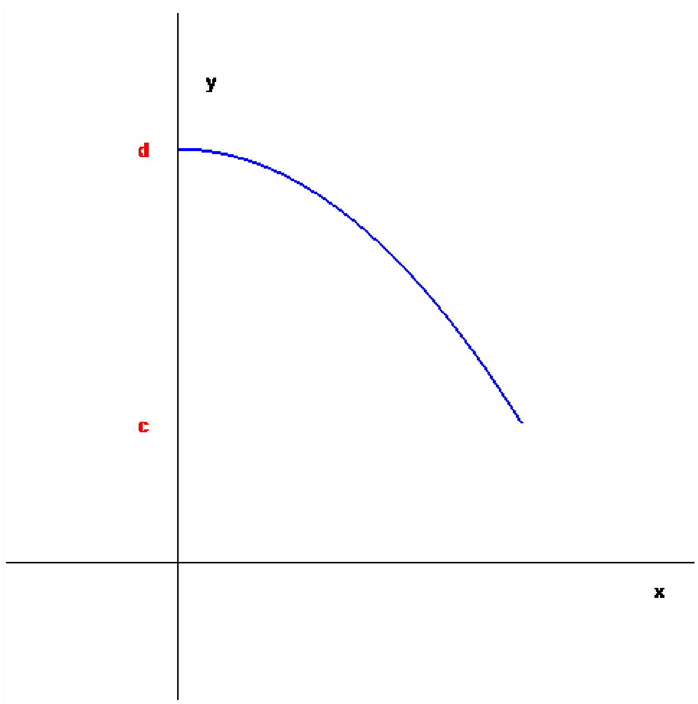
To generate the volume of the entire solid, we divide up the interval from $x = a$ to $x = b$ into n equal subintervals, for n rectangles, and add up the volumes of the disks formed by rotating each rectangle around the x axis.

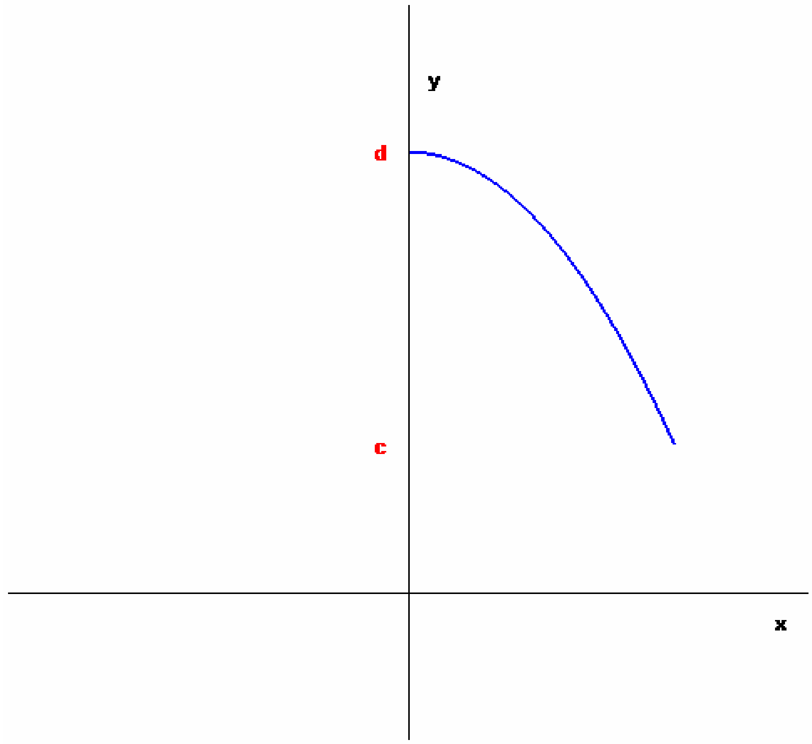
$$\text{Volume of a single Disk } i = \pi[R(x_i)]^2 \Delta x$$

$$\text{Volume of the Solid} \approx \sum_{i=1}^{n} \pi[R(x_i)]^2 \Delta x = \pi \sum_{i=1}^{n} [R(x_i)]^2 \Delta x$$

$$\text{Exact Volume of the Solid} = \lim_{n\to\infty} \pi \sum_{i=1}^{n} \left[R(x_i)\right]^2 \Delta x = \pi \int_a^b [R(x)]^2 \, dx$$

We can also rotate around the y-axis.





Exact Volume of the Solid = $\pi\int_c^d [R(y)]^2\,dy$

Disc Method

To find the volume of a solid of revolution, we use

Horizontal Axis of Revolution: $\text{Volume} = \pi\int_a^b [R(x)]^2\,dx$

Vertical Axis of Revolution: $\text{Volume} = \pi\int_c^d [R(y)]^2\,dy$

Notes

1. The representative rectangle in the disk method is always perpendicular to the axis of rotation.
2. It is important to realize that when finding the function R representing the radius of the solid, measure from the axis of revolution.

Example 1: Find the volume of the solid generated by revolving the region bounded by $y = 2x^2$, $x = 0$, and $y = 8$ about the

a. y axis.

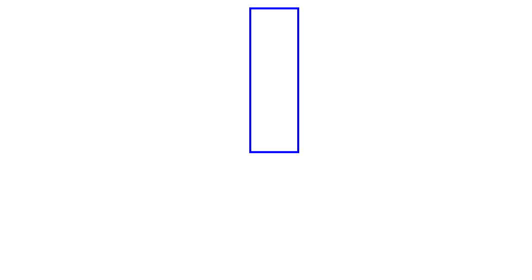
b. the line $x = 8$.

Solution:

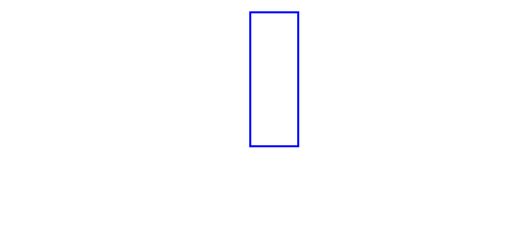
[redacted]

The Washer Method

Involves extending the disk method to cover solids of revolution with a hole. Consider the following rectangle, where *w* is the width, *R* is the distance from the axis of revolution to the outer edge of the rectangle, and *r* is the distance from the axis of revolution to the inner edge of the rectangle.

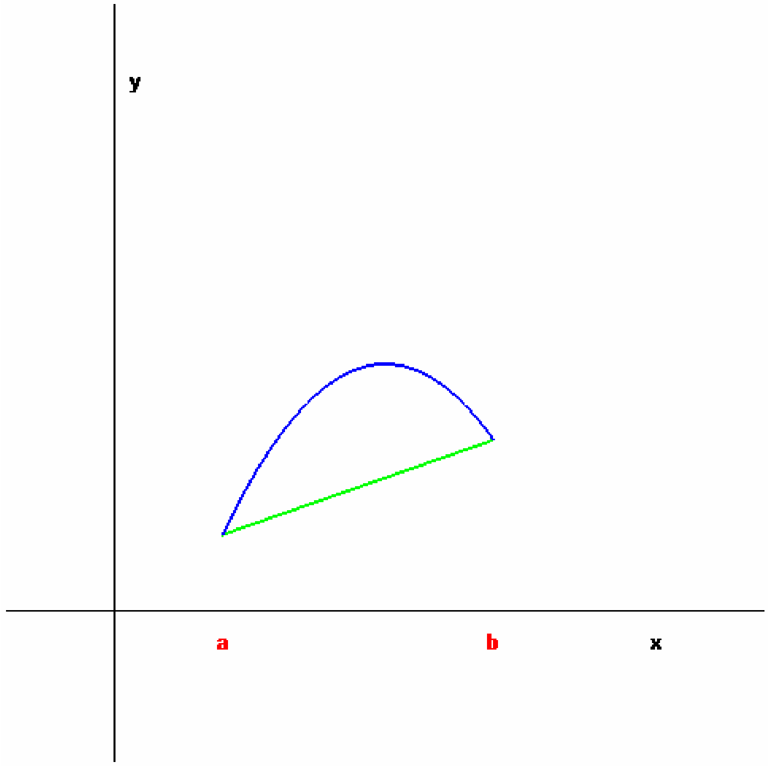


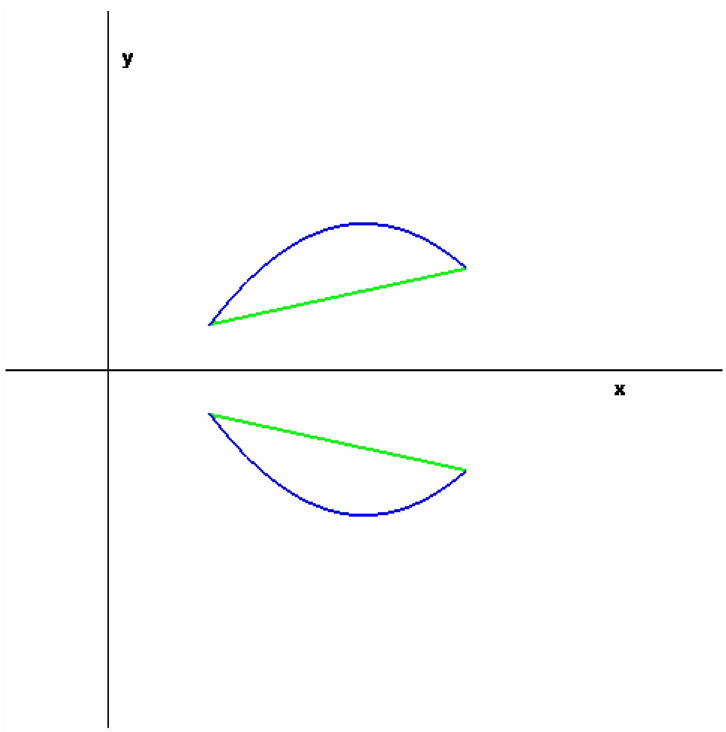
If we rotate this rectangle about this axis, we sweep out a circular washer.



Look at Figures 8, 9, and 10 p. 452

Now, consider the region bound by an outer radius $R(x)$ and inner radius $r(x)$ and suppose we rotate this region around the x axis.





Then

$$\text{Volume of Washer} = \underbrace{\pi\int_a^b [R(x)]^2\,dx}_{\text{Volume of entire solid}} - \underbrace{\pi\int_a^b [r(x)]^2\,dx}_{\text{Volume of entire hole}}$$

$$= \pi\int_a^b \left([R(x)]^2 - [r(x)]^2\right)dx$$

Example 2: Find the volume of the solid obtained by rotating the region bounded by $y = x^2$, $y^2 = x$ about the

a. y axis

b. $y = 2$.

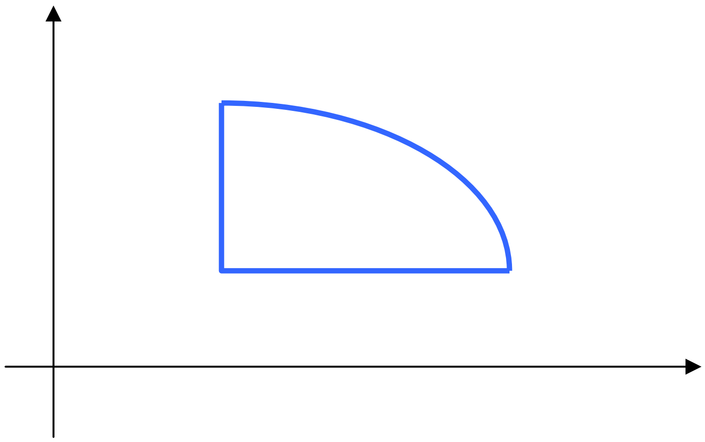
Solution:

[redacted]

The Shell Method

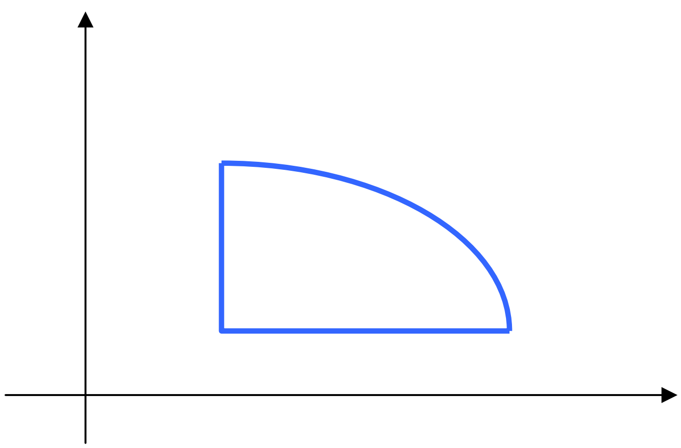
Uses cylindrical shells to find the volume of a solid of revolution.

Horizontal Axis of Revolution



$$\text{Volume} = \text{V} = 2\pi\int_c^d p(y)h(y)\,dy$$

Vertical Axis of Revolution



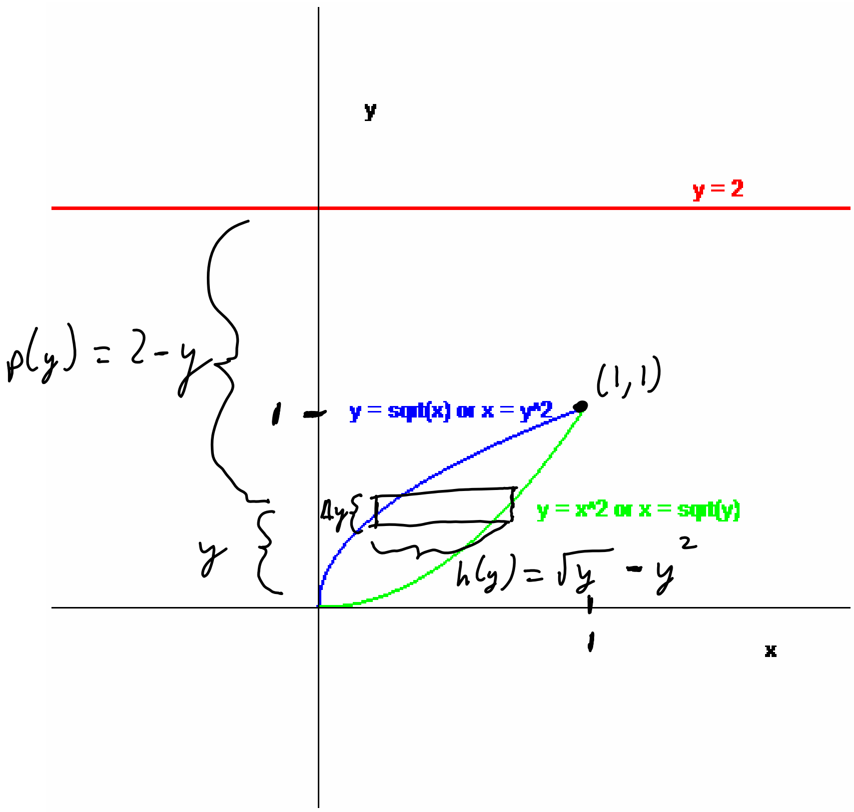
$$\text{Volume} = \text{V} = 2\pi\int_c^d p(y)h(y)\,dy$$

Notes

1. In the shell method, the representative rectangle is drawn parallel to the axis of rotation. This results in using an opposite variable of integration compared with the disk/washer method.
2. The function $p(x)$ or $p(y)$ represents the distance from the axis of rotation to the center of the representative rectangle.
3. The function $h(x)$ or $h(y)$ represents the height of the representative rectangle.

Example 3: Use the shell method to find the volume of the solid obtained by rotating the region bounded by $y = x^2$ and $y^2 = x$ about the line $y = 2$.

Solution: The following is the graph of the region bounded by the graphs of $y = x^2$ and $y^2 = x$.



Note that the representative rectangle is drawn parallel to the axis of rotation $y = 2$, which is a horizontal axis. Thus, we will integrate with respect to y. The function $h(y)$ represents the height of the representative rectangle measured from left to right. Hence, $h(y) = \sqrt{y} - y^2$. The function $p(y)$ represents the distance from the axis of rotation to the center of the representative rectangle. From the x axis, the distance to the center of the representative rectangle is the distance up the y axis, which is given by y. Hence, the distance from the axis of rotation $y = 2$ is $p(y) = 2 - y$. Thus, we say that

(Continued on next page)

$$\text{Volume} = 2\pi \int_{y=c}^{y=d} p(y)\, h(y)\, dy$$

$$= 2\pi \int_{y=0}^{y=1} (2 - y)\,(y^{\frac{1}{2}} - y^2)\, dy$$

$$= 2\pi \int_{y=0}^{y=1} (2y^{\frac{1}{2}} - 2y^2 - y^{\frac{3}{2}} - y^3)\, dy \qquad \text{(FOIL)}$$

$$= 2\pi\left(2\frac{y^{\frac{3}{2}}}{\frac{3}{2}} - 2\frac{y^3}{3} - \frac{y^{\frac{5}{2}}}{\frac{5}{2}} - \frac{y^4}{4}\right)\Bigg|_0^1 \qquad \text{(Integrate)}$$

$$= 2\pi\left(\frac{4}{3}y^{\frac{3}{2}} - \frac{2}{3}y^3 - \frac{2}{5}y^{\frac{5}{2}} - \frac{y^4}{4}\right)\Bigg|_0^1$$

$$= 2\pi\left(\frac{4}{3}(1)^{\frac{3}{2}} - \frac{2}{3}(1)^3 - \frac{2}{5}(1)^{\frac{5}{2}} - \frac{(1)^4}{4}\right) - 0$$

$$= 2\pi\left(\frac{4}{3} - \frac{2}{3} - \frac{2}{5} - \frac{1}{4}\right)$$

$$= 2\pi\left(\frac{2}{3} - \frac{2}{5} - \frac{1}{4}\right)$$

$$= 2\pi\left(\frac{40}{60} - \frac{24}{60} - \frac{15}{60}\right) \qquad \text{(Common Denominator is 60)}$$

$$= 2\pi\left(\frac{31}{60}\right)$$

$$\boxed{= \frac{31\pi}{30}}$$

$\blacksquare$