Section 7.2: Direction Fields and Euler's Methods

Practice HW from Stewart Textbook (not to hand in)
p. 511 # 1-13, 19-23 odd

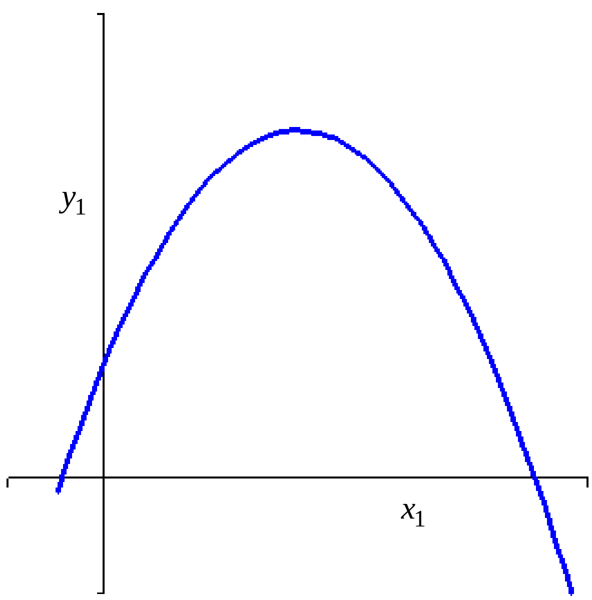
For a given differential equation, we want to look at ways to find its solution. In this chapter, we will examine 3 techniques for determining the behavior for the solution. These techniques will involve looking at the solutions graphically, numerically, and analytically.

Examining Solutions Graphically – Direction Fields

Recall from Calculus I that for a function $y(x)$, $y' = \frac{dy}{dx}$ gives the <u>slope</u> of the tangent line at a particular point (x, y) on the graph of $y(x)$. Suppose we consider a first order differential equations of the form

$$y' = f(x, y).$$

For a solution y of this differential equation, y' evaluated at the point (x_1, y_1) represents the slope of the tangent line to the graph of $y(x)$ at this point.



Even though we do not know the formula for the solution $y(x)$, having the differential equation $y' = f(x, y)$ gives a convenient way for calculating the tangent line slopes at various points. If we obtain these slopes for many points, we can get a good general idea of how the solution is behaving.

Direction Fields (sometimes called *slope fields*) involves a method for determining the behavior of various solutions on the *x*-*y* plane by calculating the tangent line slopes at various points.

Example 1: Sketch the direction field for the differential equation $y' = x^2 - y$. Use the result to sketch the graph of the solution with initial condition $y(0) = 1$.

Solution: In this problem, we plot points for the four quadrant regions and the *x* and *y* axis (we will fill in the first quadrant chart in class).

1st Quadrant		
x	y	$y' = x^2 - y$
1	1	
2	1	
3	1	
1	2	
2	2	
3	2	
1	3	
2	3	
3	3	

2nd Quadrant		
x	y	$y' = x^2 - y$
-3	1	8
-2	1	3
-1	1	0
-3	2	7
-2	2	2
-1	2	-1
-3	3	6
-2	3	1
-1	3	-2

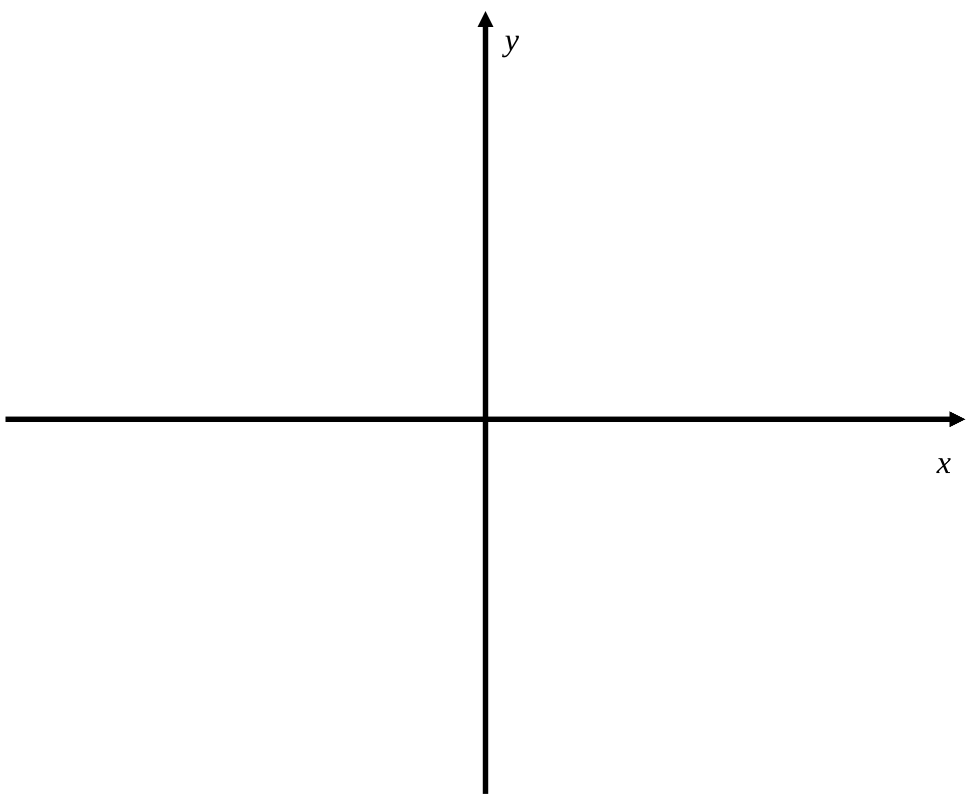
3rd Quadrant		
x	y	$y' = x^2 - y$
-3	-1	10
-2	-1	5
-1	-1	2
-3	-2	11
-2	-2	6
-1	-2	3
-3	-3	12
-2	-3	7
-1	-3	4

4th Quadrant		
x	y	$y' = x^2 - y$
1	-1	2
2	-1	5
3	-1	10
1	-2	3
2	-2	6
3	-2	11
1	-3	4
2	-3	7
3	-3	12

***x*-axis**		
x	y	$y' = x^2 - y$
-3	0	9
-2	0	4
-1	0	1
0	0	0
1	0	1
2	0	4
3	0	9

***y*-axis**		
x	y	$y' = x^2 - y$
0	-3	3
0	-2	2
0	-1	1
0	0	0
0	1	-1
0	2	-2
0	3	3

We can sketch the slopes on the following graph (will do in class):

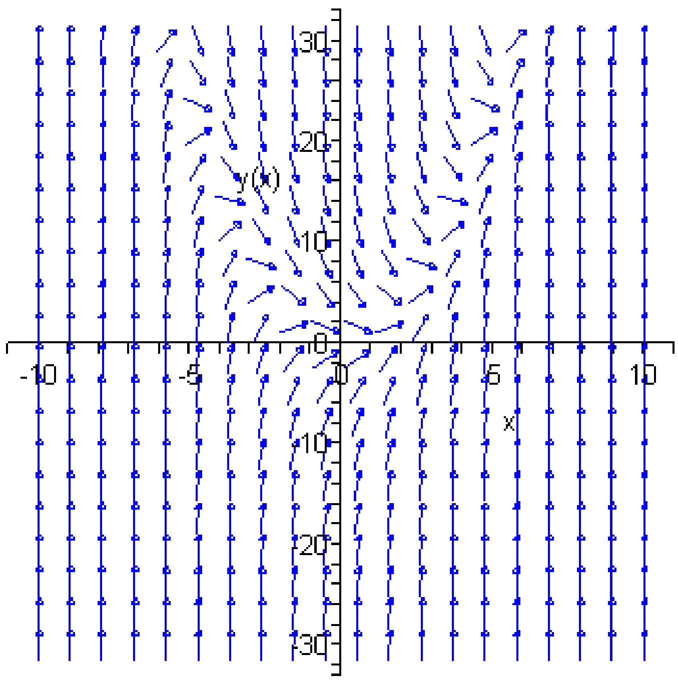


Obviously, as can be seen by the last example, sketching direction fields by hand can be a very tedious task. However, Maple can sketch a direction field quickly. For the differential equation $y' = x^2 - y$ given in Example 1, the following commands in Maple can be used to sketch the direction field:

```
> with(DEtools): with(plots):
Warning, the name changecoords has been redefined
> de := diff(y(x),x)=x^2-y(x);
```

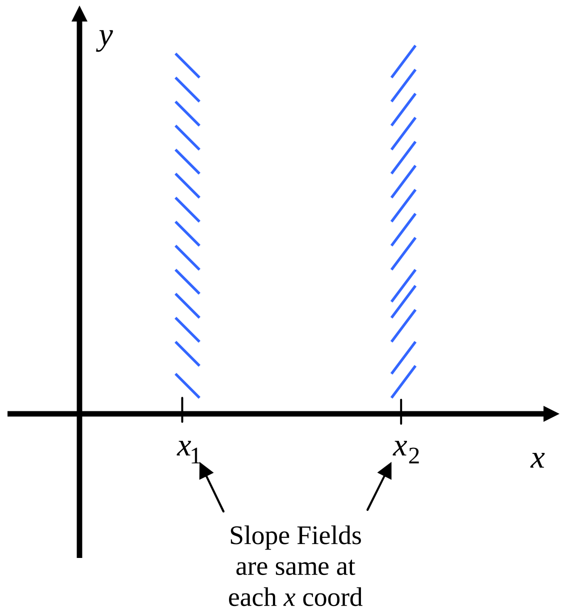
$$de := \frac{d}{dx}\, y(x) = x^2 - y(x)$$

```
> dfieldplot(de, y(x),x=-10..10,y=-30..30, color = black,
arrows = MEDIUM, color = blue);
```

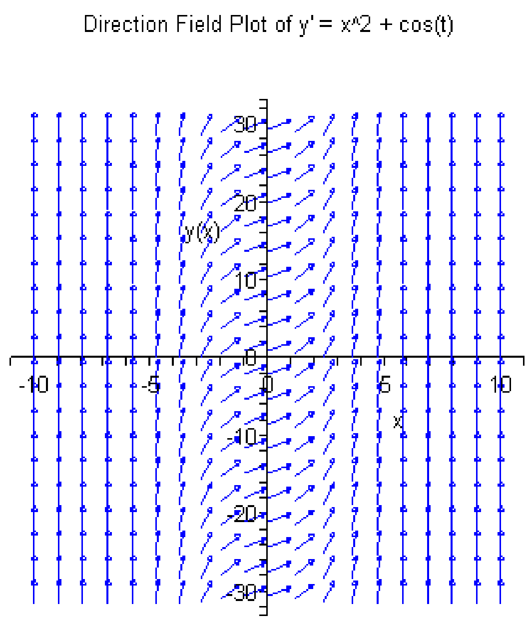


Notes

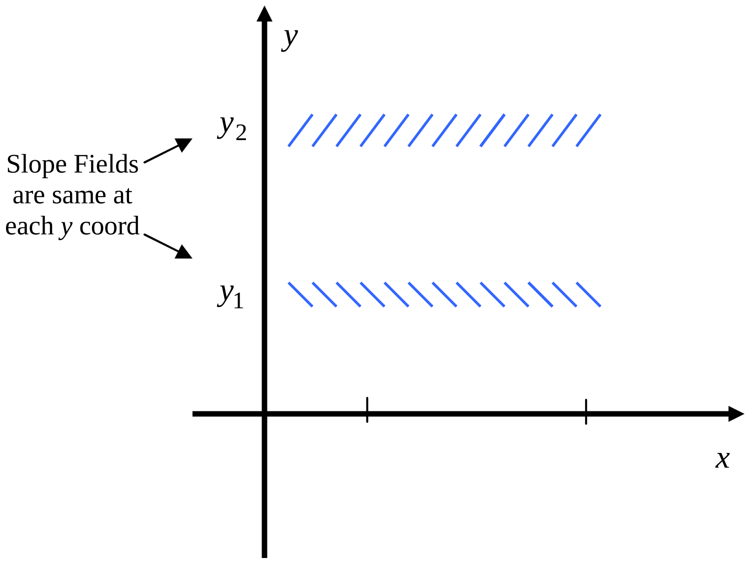
1. The direction fields for differential equations of the form $y' = f(x)$, where the right is strictly a function of *x* have the same slope fields for points with the same *x* coordinate.



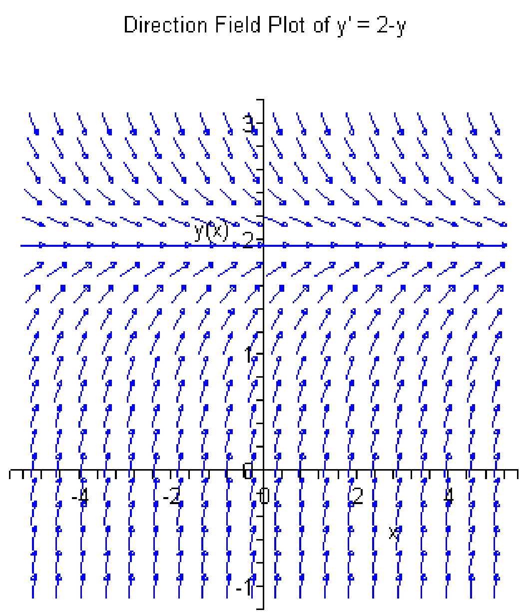
Example: Plot of $y' = t^2 + \cos(t)$



2. 1The direction fields for differential equations of the form $y' = f(y)$, where the right is strictly a function of *y* have the same slope fields for points with the same *y* coordinate. A differential equation is strictly a function of the dependent variable *y* is known as an *autonomous* equation.



Example: Plot of $y' = (2 - y)$

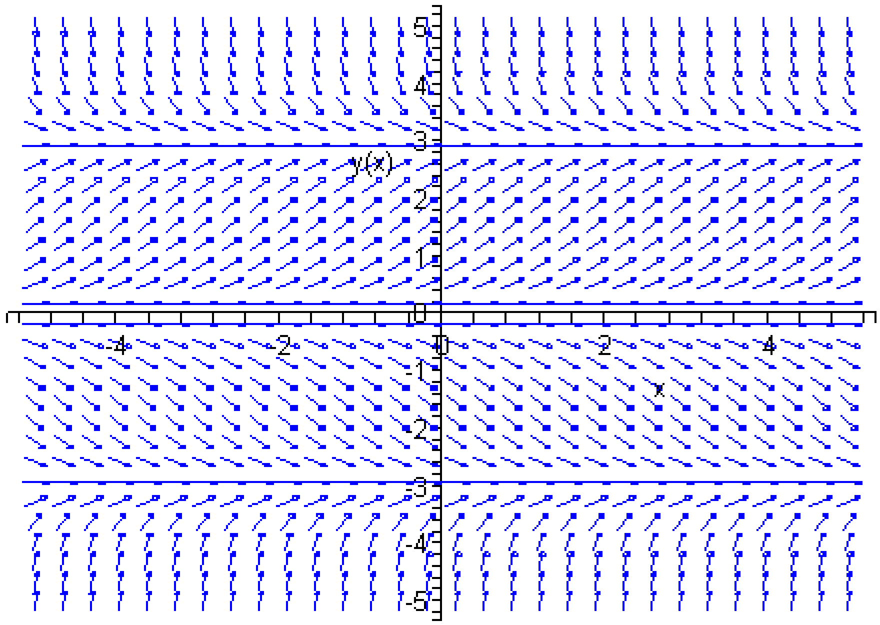


3. A constant solution of the form $y = K$ of an autonomous where the direction field slopes are zero, that is, where $y' = 0$ and the solution *y* neither increases or decreases, is known as an equilibrium solution.

Example: $y' = (2 - y)$ the equilibrium solution is $y = 2$.

Example 2: Given the direction field plot of the differential equation $y' = y(1 - y^2/9)$.

Direction Field Plot of y' = y(1-y^2/9)



a. Sketch the graphs of solutions that satisfy the given initial conditions:
   i. $y(0) = 1$
   ii. $y(0) = 3$
   iii. $y(0) = -2$
b. Find all equilibrium solutions.

Solution:

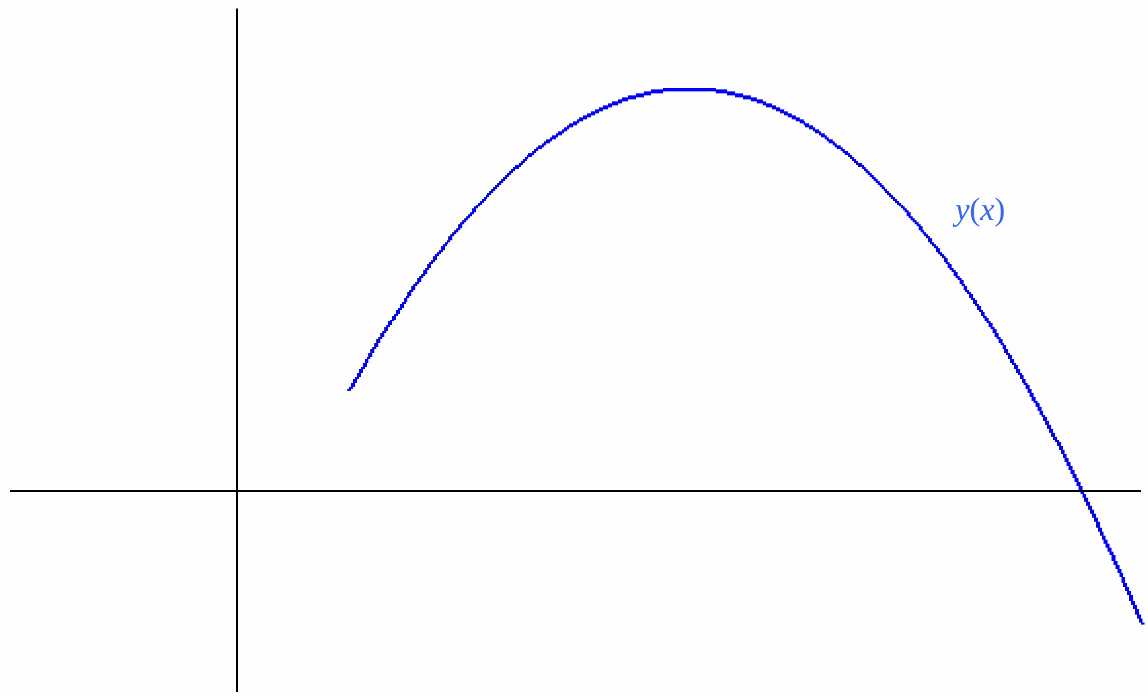
Finding Solutions Numerically – Euler's Method

A common way to examine the solution of a differential equations is to approximate it numerically. One of the more simpler methods for doing this involves Euler's method.

Consider the initial value problem

$$y' = F(x, y), \;\; y(x_0) = y_0 .$$

over the interval $x_0 = a \le x \le b$. Suppose we want to find an approximation to the solution $y(x)$ given by the following graph:



Starting at the point (x_0, y_0) specified by the initial condition $y(x_0) = y_0$, we want to approximate to solution at equally spaced points beyond x_0 on the x axis. Let h (known as the step size) be the space between the points on the x-axis. Then $x_1 = x_0 + h$, $x_2 = x_1 + h$, $x_3 = x_2 + h$, etc. Consider the tangent line at the point (x_0, y_0) that passes through the point (x_1, y_1). Since the derivative is used to calculate the slope of the tangent line, it can be seen that

$$\begin{matrix}\text{Slope of the tangent line} \\ \text{to } y(x) \text{ at } (x_0, y_0)\end{matrix} = y'\big|_{(x_0, y_0)} = F(x_0, y_0)$$

Hence,

$$\begin{aligned}
\begin{matrix}\text{Slope through}\\ (x_0, y_0) \text{ and } (x_1, y_1)\end{matrix} &= \text{Slope at tangent line at } (x_0, y_0) \\
\frac{y_1 - y_0}{x_1 - x_0} &= F(x_0, y_0) \\
y_1 - y_0 &= (x_1 - x_0)\, F(x_0, y_0) \\
y_1 - y_0 &= h\; F(x_0, y_0) \\
y_1 &= y_0 + h\; F(x_0, y_0)
\end{aligned}$$

Now, consider the line through the points (x_1, y_1) and (x_2, y_2).

$$\begin{aligned}
\begin{matrix}\text{Slope through}\\ (x_1, y_1) \text{ and } (x_2, y_2)\end{matrix} &\approx \text{Slope at tangent line at } (x_1, y_1) = F(x_1, y_1) \\
\frac{y_2 - y_1}{x_2 - x_1} &= F(x_1, y_1) \\
y_2 &= y_1 + h\; F(x_1, y_1)
\end{aligned}$$

In general,

$$y_n = y_{n-1} + h\; F(x_{n-1}, y_{n-1})$$

Summarizing,

Euler's Method

Given the initial value problem

$$y' = F(x, y), \;\; y(x_0) = y_0$$

we calculate (x_n, y_n) from (x_{n-1}, y_{n-1}) by computing

$$x_n = x_{n-1} + h$$

$$y_n = y_{n-1} + h\; F(x_{n-1}, y_{n-1})$$

where h is the step size between endpoints on the x-axis.

Example 3: Use Euler's Method with step size of 0.5 to estimate $y(2)$, where $y(x)$ is the solution to the initial value problem $y' = 2x - 3y$, $y(0) = 4$. Sketch the graph of the iterates used in find the estimate.

Solution:

Notes

1. Using techniques that can studied in a differential equations course, it can be shown that the exact solution to the initial value problem $y' = 2x - 3y,\ y(0) = 4$ given in Example 3 is

$$y(x) = \frac{38}{9}e^{-3x} + \frac{2}{3}x - \frac{2}{9}$$

The approximation to $y(2)$ (what y is when $x = 2$) was $y_4 = 1.375$. The exact value is $y(2) = \frac{38}{9}e^{-3(2)} + \frac{2}{3}(2) - \frac{2}{9} = \frac{38}{9}e^{-6} + \frac{4}{3} - \frac{2}{9} \approx 1.1215769.$ Thus the error between the approximation and the exact value is

$$|\, y(2) - y_4 \,| = |\, 1.1215769 - 1.375 \,| \approx |\, -0.253423 \,| = 0.253423.$$

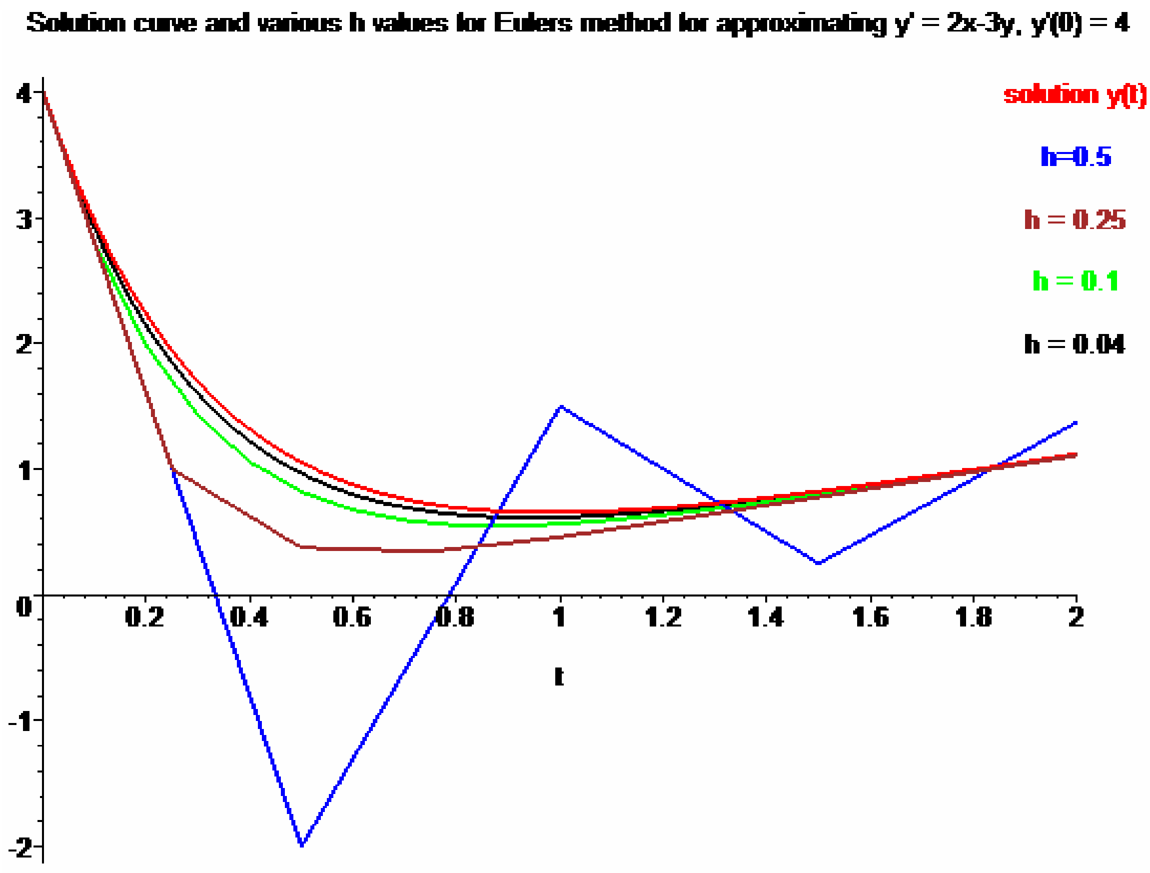
2. By decreasing the step size h, the accuracy of the approximation in most cases will be better, with a tradeoff in more work needed to achieve the approximations. For example, the chart below shows the approximations generated when the step size for Example 3 is cut in half to h = 0.25.

$x_n = x_{n-1} + h$ $= x_{n-1} + 0.25$	$y_n = y_{n-1} + h\ F(x_{n-1}, y_{n-1})$ $= y_{n-1} + 0.25(2x_{n-1} - 3y_{n-1})$	Exact Value $y(x) = \frac{38}{9}e^{-3x} + \frac{2}{3}x - \frac{2}{9}$
$x_0 = 0$	$y_0 = 4$	$y(0) = \frac{38}{9}e^{0} + \frac{2}{3}(0) - \frac{2}{9} = \frac{36}{9} = 4$
$x_1 = x_0 + h$ $= 0 + 0.25$ $= 0.25$	$y_1 = y_0 + 0.25(2x_0 - 3y_0)$ $= 4 + 0.25(2(0) - 3(4))$ $= 4 + 0.25(0 - 12) = 4 + (-3) = 1$	$y(0.25) = \frac{38}{9}e^{-3(0.25)} + \frac{2}{3}(-0.25) - \frac{2}{9}$ $\approx 1.938881000;$
$x_2 = 0.5$	$y_2 = 0.375$	$y(0.5) \approx 1.053216232$
$x_3 = 0.75$	$y_3 = 0.34375$	$y(0.75) \approx 0.7227967261$
$x_4 = 1$	$y_4 = 0.4609375$	$y(1) \approx 0.6546565109$
$x_5 = 1.25$	$y_5 = 0.6152343750$	$y(1.25) \approx 0.7104082603$
$x_6 = 1.5$	$y_6 = 0.7788085938$	$y(1.5) \approx 0.8246824299$
$x_7 = 1.75$	$y_7 = 0.9447021484$	$y(1.75) \approx 0.9666006336$
$x_8 = 2$	$y_8 = 1.111175537$	$y(2) \approx 1.1215769542$

Here, the approximation to $y(2) \approx 1.1215769542$ is $y_8 = 1.111175537$ and the the error between the approximation and the exact value is

$$|\, y(2) - y_8 \,| \approx |\, 1.1215769 - 1.1111755 \,| \approx |\, 0.010401 \,| = 0.010401.$$

The following represents a graph of the curves produced by Euler's method for various values of *h* and the exact solution *y*(*x*).



3. There are other numerical methods that can achieve better accuracy with less work than Euler's method. However, the underlying approach used in many of these methods stem from Euler's approach.