Section 8.6/8.7: Taylor and Maclaurin Series

Practice HW from Stewart Textbook (not to hand in)
p. 604 # 3-15 odd, 21-27 odd
p. 615 # 5-25 odd, 31-37 odd

Taylor Series

In this section, we discuss how to use a power series to represent a function.

Definition: If $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ has a power series representation, then $c_n = \dfrac{f^n(x)}{n!}$
and

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(x)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \ldots + \frac{f^n(a)}{n!}(x-a)^n + \ldots$$

which is called a *Taylor series* at $x = a$.

If $a = 0$, then

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(x)}{n!}x^n = f(0) + f'(0)\,x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots + \frac{f^n(0)}{n!}x^n + \ldots$$

is the Maclaurin series of f centered at $x = 0$.

Example 1: Find the Maclaurin series of the function $f(x) = e^x$. Find the radius of convergence of this series.

Solution:

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Example 2: Find the Maclaurin series of the function $f(x) = \dfrac{1}{1-x}$. Find the radius of convergence of this series.

Solution: Since the Maclaurin series is a special case of a Taylor series centered at $a = 0$, its formula is given by

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{n}(x)}{n!} x^n = f(0) + f'(0)\,x + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} x^3 + \frac{f^{4}(0)}{4!} x^4 + \ldots$$

Noting by the exponent law $b^{-m} = \dfrac{1}{b^m}$ that $f(x) = \dfrac{1}{1-x} = (1-x)^{-1}$, we obtain the following terms for this formula.

$$f(x) = (1-x)^{-1} = \frac{1}{1-x} \Rightarrow f(0) = \frac{1}{1-0} = \frac{1}{1} = 1.$$

$$f'(x) = \underbrace{(-1)(1-x)^{-2}(0-1)}_{\text{Use Chain (General Power) Rule}} = (-1)(1-x)^{-2}(-1) = (1-x)^{-2} = \frac{1}{(1-x)^2} \Rightarrow f'(0) = \frac{1}{(1-0)^2} = \frac{1}{(1)^2} = \frac{1}{1} = 1$$

$$f''(x) = \underbrace{(-2)(1-x)^{-3}(-1)}_{\text{Use Chain (General Power) Rule}} = 2(1-x)^{-3} = \frac{2}{(1-x)^3} \Rightarrow f''(0) = \frac{2}{(1-0)^3} = \frac{2}{(1)^3} = \frac{2}{1} = 2$$

$$f'''(x) = \underbrace{2(-3)(1-x)^{-4}(-1)}_{\text{Use Chain (General Power) Rule}} = 6(1-x)^{-4} = \frac{6}{(1-x)^4} \Rightarrow f'''(0) = \frac{6}{(1-0)^4} = \frac{6}{(1)^4} = \frac{6}{1} = 6$$

$$f^{4}(x) = \underbrace{6(-4)(1-x)^{-5}(-1)}_{\text{Use Chain (General Power) Rule}} = 24(1-x)^{-5} = \frac{24}{(1-x)^5} \Rightarrow f^{4}(0) = \frac{24}{(1-0)^5} = \frac{24}{(1)^5} = \frac{24}{1} = 24$$

Hence,

$$f(x) = \frac{1}{1-x} = f(0) + f'(0)x + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} x^3 + \frac{f^{4}(0)}{4!} x^4 + \ldots$$

$$= (1) + (1)\,x + \frac{2}{2} x^2 + \frac{6}{6} x^3 + \frac{24}{24} x^4 + \ldots \quad \text{(Substitute in values)}$$

$$= 1 + x + x^2 + x^3 + x^4 + \ldots \quad \text{(Simplify)}$$

$$= \sum_{n=0}^{\infty} x^n$$

(continued on next page)

To determine the interval of convergence for this series, we use the ratio test. If we set $a_n = x^n$, then we have $a_{n+1} = x^{n+1}$. Hence, by the ratio test we have

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{x^n}\right| = \lim_{n\to\infty}\left|\frac{\cancel{x^n}\cdot x^1}{\cancel{x^n}}\right| = \lim_{n\to\infty}|x| = |x|$$

For convergence to occur, $|x| < 1$. Hence, by definition of absolute value , the initial interval of convergence is $-1 < x < 1$. We next test possible convergence at the endpoint so this interval, x = -1 and x = 1. Using the series formula for our answer $f(x) = \sum_{n=0}^{\infty} x^n$, we have

$$x = -1 \Rightarrow \sum_{n=0}^{\infty}(-1)^n = 1-1+1-1+1-1+\ldots \qquad \text{(Note } (-1)^0 = 1,\ (-1)^1 = -1,\ (-1)^2 = 1, \text{ etc.)}$$

This series is an alternating series, but an easy way to test its convergence is to note it is a geometric series with $r = -1$. Since $|r| = |-1| = 1 \geq 1$, the series is divergent. For the other endpoint, we have

$$x = 1 \Rightarrow \sum_{n=0}^{\infty}(1)^n = 1+1+1+1+1-1+\ldots$$

$$= \sum_{n=1}^{\infty} n$$

For the sequence formula $a_n = n$, since $\lim_{n\to\infty} n = \infty > 0$, the sequence convergence tests says the series diverges. Hence, the series diverges at both endpoints x = -1 and x = 1. Thus, the interval of convergence is

$$\boxed{-1 < x < 1.}$$

$\blacksquare$

Note: Once we know the Maclaurin (Power) series representations centered at $x = 0$ for a given function, we can find the Maclaurin (Power) series of other functions by substitution, differentiation, or integration.

Some Common Maclaurin Series

Series	**Interval of Convergence**
$\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \ldots$	$-1 < x < 1$
$e^x = \sum_{n=0}^{\infty} \dfrac{x^n}{n!} = 1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!} + \ldots$	$-\infty < x < \infty$
$\sin x = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{(2n+1)!} = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \ldots$	$-\infty < x < \infty$
$\cos x = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n}}{(2n)!} = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{6!} - \dfrac{x^8}{8!} + \ldots$	$-\infty < x < \infty$
$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{2n+1} = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \ldots$	$-1 \le x \le 1$

Example 3: Use a known Maclaurin series to find the Maclaurin series of the given function $f(x) = \sin x^4$.

Solution:

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Example 4: Use a known Maclaurin series to find the Maclaurin series of the given function $f(x) = xe^{2x}$.

Solution:

■

Note: To differentiate or integrate a Maclaurin or Taylor series, we differentiate or integrate term by term.

Example 5: Find the Maclaurin series of $\int \sin x^4 dx$.

Solution:

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Example 6: Use a series to estimate $\int_0^1 \sin x^4 dx$ to 3 decimal places.

Solution: From the previous exercise, we saw that

$$\int \sin x^4 dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{8n+5}}{(2n+1)!(8n+5)} + C .$$

Then

$$\begin{aligned}\int_0^1 \sin x^4 dx &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{8n+5}}{(2n+1)!(8n+5)}\Bigg|_{x=0}^{x=1} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n (1)^{8n+5}}{(2n+1)!(8n+5)} \quad - \quad 0 \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!(8n+5)} \\ &= \frac{1}{5} - \frac{1}{3!\cdot 13} + \frac{1}{5!\cdot 21} - \frac{1}{7!\cdot 29} + \ldots\end{aligned}$$

Use the alternating series estimate theorem we saw in Section 8.4, we would like the value of the sequence term n where $b_n = \dfrac{1}{(2n+1)!\,(8n+5)} < 0.0009$. The following Maple commands illustrate that this occurs when n = 2.

```
> b := n -> 1/(factorial(2*n+1)*(8*n+5));
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$$b := n \rightarrow \frac{1}{(2\,n+1)!\,(8\,n+5)}$$

```
> evalf(b(0));
```

0.2000000000

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> evalf(b(1));
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0.01282051282

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> evalf(b(2));
```

0.0003968253968

Since the error is computed using the term b_2 in the series, the estimate can be computed by summing the terms in the series, b_0 and b_1, that precede it. Thus

$$\int_0^1 \sin(x^4)\, dx \approx \frac{1}{5} - \frac{1}{3!\cdot 13} = \frac{1}{5} - \frac{1}{78} \boxed{\approx 0.187179}$$

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Example 7: Find the Taylor series of $f(x) = \sin x$ at $a = \frac{\pi}{4}$.

Solution:

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Example 8: Find the Taylor series of $f(x) = \ln x$ at $a = 2$.

Solution:

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