Section 8.9: Applications of Taylor Polynomials

Practice HW from Stewart Textbook (not to hand in)
p. 628 # 1-21 odd

Taylor Polynomials

In this section, we use Taylor polynomials to approximate a given function $f(x)$ near a point $x = a$.

Definition: The n^{th} Taylor polynomial $T_n(x)$ at a function $f(x)$ at $x = a$ is given by

$$T_{n(}(x) = f(a) + f'(a)\frac{f'(a)}{1!}(x-a)\frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \ldots + \frac{f^n(a)}{n!}(x-a)^n$$

where $T_n(a) = f(a), \quad T_n'(a) = f'(a), \quad T_n''(a) = f''(a), \; \ldots, \quad T_n^n(a) = f^n(a)$.

Example 1: Find the Taylor (Maclaurin) polynomial for $f(x) = e^x$ at $a = 0$ and $n = 2$ and use it to approximate $e^{0.01}$.

Solution:

Note: In general, the higher the degree of the Taylor polynomial and the closer we are to the point $x = a$ that the Taylor polynomial is centered at, the better the approximation

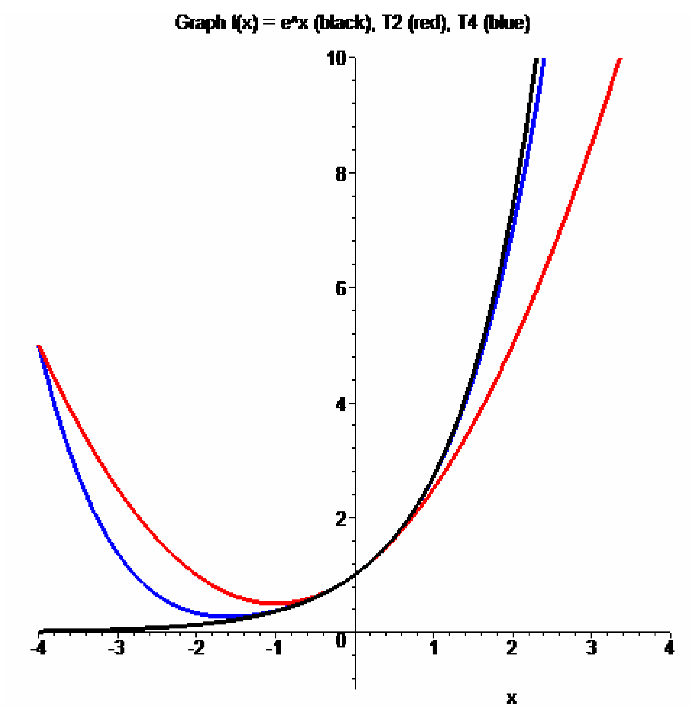
Example 2: Find the Taylor (Maclaurin) polynomial for $f(x) = e^x$ at $a = 0$ and $n = 4$ and use it to approximate $e^{0.01}$.

Solution:

The following table illustrates the accuracy of the Taylor polynomials $T_2(x) = 1 + x + \frac{1}{2}x^2$ and $T_4(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4$ to $f(x) = e^x$ near $x = 0$.

x	-8	-1	-0.1	-0.01	0	0.01	0.1	1	5
$f(x) = e^x$	0.0003	0.367879	.904837	0.9900498	1	1.0100501	1.1051709	2.7182818	148.413
$T_2(x)$	25	0.5	0.905	0.99005	1	1.01005	1.105	2.5	18.5
$T_4(x)$	110.3	0.375	0.904837	0.9900498	1	1.0100501	1.1051708	2.7083333	65.375

As can be seen, the closer the value of x is nearer to zero, the better the approximation the Taylor polynomials provided for the function. This is further illustrated by the following graphs:



Note: The following Maple commands can be used to find the 2nd and 4th degree Taylor polynomials for $f(x) = e^x$ near $x = 0$.

```
> with(Student[Calculus1]):
> TaylorApproximation(exp(x), x = 0, order = 2);
```

$$1 + x + \frac{1}{2}x^2$$

```
> TaylorApproximation(exp(x), x = 0, order = 4);
```

$$1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4$$

Example 3: Find the Taylor polynomial $T_n(x)$ for the function $f(x) = \sqrt{3 + x^2}$ at $a = 1$ for $n = 3$.

Solution:

Error of Approximation

Can be used to determine how close a Taylor polynomial $T_n(x)$ is to $f(x)$.

We define the function $R_n(x)$, which represents the error between $f(x)$ and $T_n(x)$, as follows:

$$R_n(x) = f(x) - T_n(x)$$

Estimation of Error – Taylor's Inequality

If $\left|f^{n+1}(x)\right| \le M$ for $|x-a| \le d$ ($-d+a \le x \le d+a$), then

$$|R_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1} \quad \text{for } |x-a| \le d$$

Note: It is many times useful to use Maple to help determine the error of approximation.

Example 4: For $f(x) = \ln(1+2x)$, use Maple to

a. Approximate f by a Taylor polynomial of degree n = 3 centered at a = 1.

b. Use Taylor's inequality to estimate the accuracy of the approximation $f(x) \approx T_n(x)$ f for $0.5 \le x \le 1.5$.

Solution (Part a): The following commands will compute and store the 3rd degree Taylor polynomial centered at $x = a = 1$.

```
> with(Student[Calculus1]):
> T3 := TaylorApproximation(ln(1+2*x), x = 1, order = 3);
```

$$T3 := \ln(3) + \frac{38\,x}{27} - \frac{80}{81} - \frac{14\,x^2}{27} + \frac{8\,x^3}{81}$$

Thus,

$$\boxed{T_3(x) = \ln(3) - \frac{80}{81} + \frac{38}{27}x - \frac{14}{27}x^2 + \frac{8}{81}x^3}$$

Solution (Part b): It is important to note that the inequality $|x-1| \le 0.5$ by definition says that $-0.5 \le x - 1 \le 0.5$. Adding 1 to all sides of the inequality gives $0.5 \le x \le 1.5$, which is the given interval. The Taylor inequality estimate says that if

$$\left|f^{n+1}(x)\right| \le M \text{, then} |R_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1} \quad \text{for } |x-a| \le d$$

For this problem, n = 3, a = 1, and d = 0.5. Thus, the equality becomes

$$|R_3(x)| \le \frac{M}{4!}|x-1|^4 \quad \text{for } |x-1| \le 0.5$$

This inequality is guaranteed to be true if $\left|f^{n+1}(x)\right| \le M$, or $\left|f^4(x)\right| \le M$ for $0.5 \le x \le 1.5$. Our goal is to find an upper bound for $\left|f^4(x)\right|$ on the interval $0.5 \le x \le 1.5$, that is, a value for which $\left|f^4(x)\right|$ is guaranteed to be smaller than for the entire interval. We can use a graph of $\left|f^4(x)\right|$ to find M. The following Maple commands can be used.

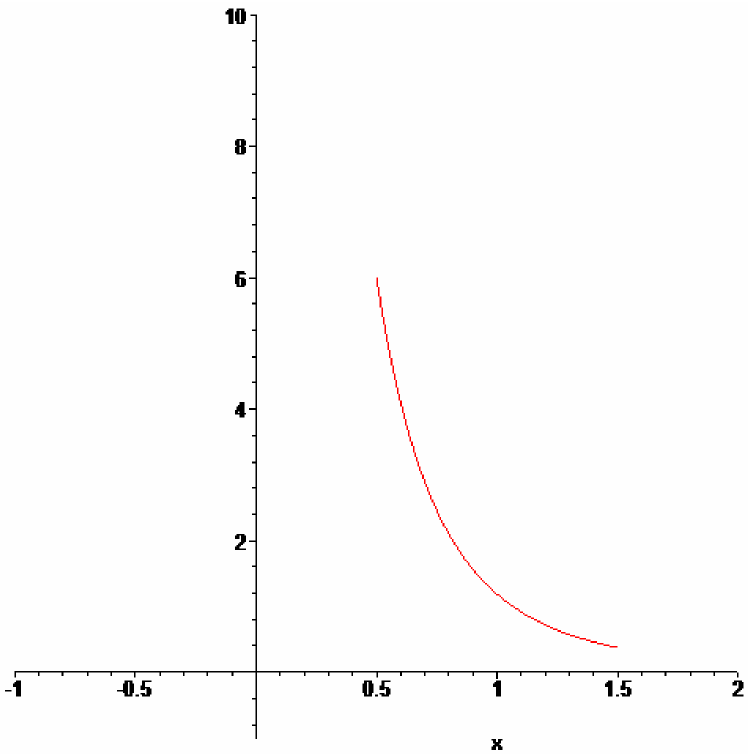
```
> f := ln(1+2*x);
```

$$f := \ln(1 + 2\,x)$$

```
> f4 := abs(diff(f, x$4));
```

$$f4 := \frac{96}{|1 + 2\,x|^4}$$

```
> plot(f4, x = 0.5..1.5, view = [-1..2, -1..10]);
```



As, the graph shows, the fourth order derivative $\left|f^{4}(x)\right|$ has a maximum of 6 on the interval $0.5 \le x \le 1.5$ and this maximum occurs at $x = 0.5$. Thus, M = 6 and $\left|f^{4}(x)\right| \le 6$ when $x = 0.5$. Hence,

$$|R_3(x)| \le \frac{M}{4!}|x-1|^4 = \frac{6}{24}|0.5-1|^4 = \frac{1}{4}|-0.5|^4 = (0.25)(0.5)^4 = (0.25)(0.0625) = 0.015625$$

Thus, this says for any value of x in the interval $0.5 \le x \le 1.5$, the function $f(x) = \ln(1+2x)$ and the Taylor polynomial $T_3(x) = \ln(3) - \frac{80}{81} + \frac{38}{27}x - \frac{14}{27}x^2 + \frac{8}{81}x^3$ will never differ more than 0.015625

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