Section 9.2: Vectors

Practice HW from Stewart Textbook (not to hand in)
p. 649 # 7-20

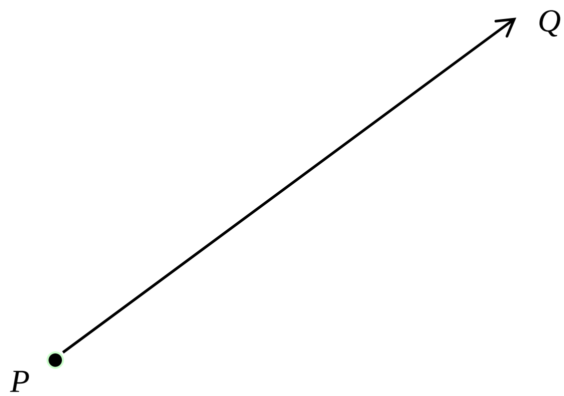
Vectors in 2D and 3D Space

Scalars are real numbers used to denote the amount (magnitude) of a quantity. Examples include temperature, time, and area.

Vectors are used to indicate both magnitude and direction. The force put on an object or the velocity a pitcher throws a baseball are examples.

Notation for Vectors

Suppose we draw a directed line segment between the points *P* (called the initial point) and the point *Q* (called the terminal point).



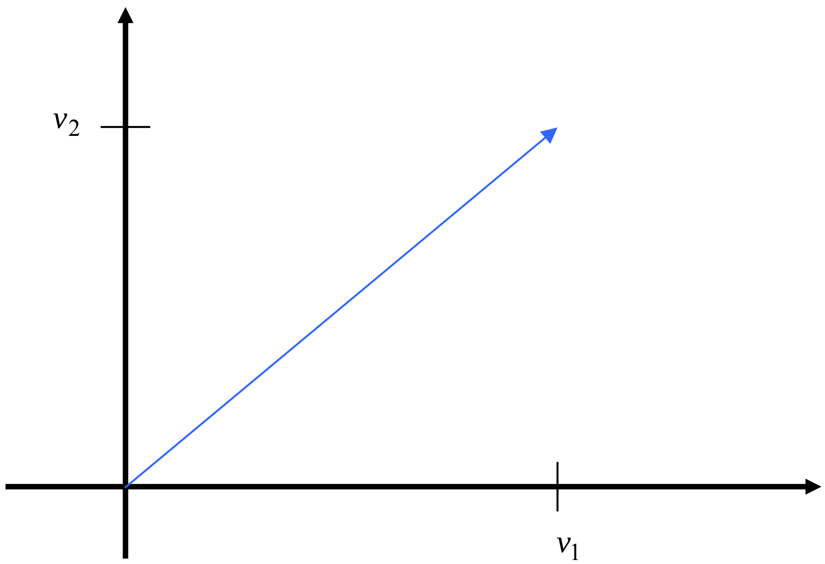
We denote the vector between the points *P* and *Q* as $\boldsymbol{v} = \overrightarrow{\boldsymbol{PQ}}$. We denote the length or magnitude of this vector as

$$\text{Length of } \boldsymbol{v} = |\boldsymbol{v}| = \left|\overrightarrow{\boldsymbol{PQ}}\right|$$

We would like a way of measuring the magnitude and direction of a vector. To do this, we will example vectors both in the 2D and 3D coordinate planes.

Vectors in 2D Space

Consider the *x*-*y* coordinate plane. In 2D, suppose we are given a vector ***v*** with initial point at the origin (0, 0) and terminal point given by the ordered pair (v_1, v_2).



The vector ***v*** with initial point at the origin (0, 0) is said to be in *standard position*. The component for of ***v*** is given by $\boldsymbol{v} = < v_1, v_2 >$.

Example 1: Write in component form and sketch the vector in standard position with terminal point (1, 2).

Solution:

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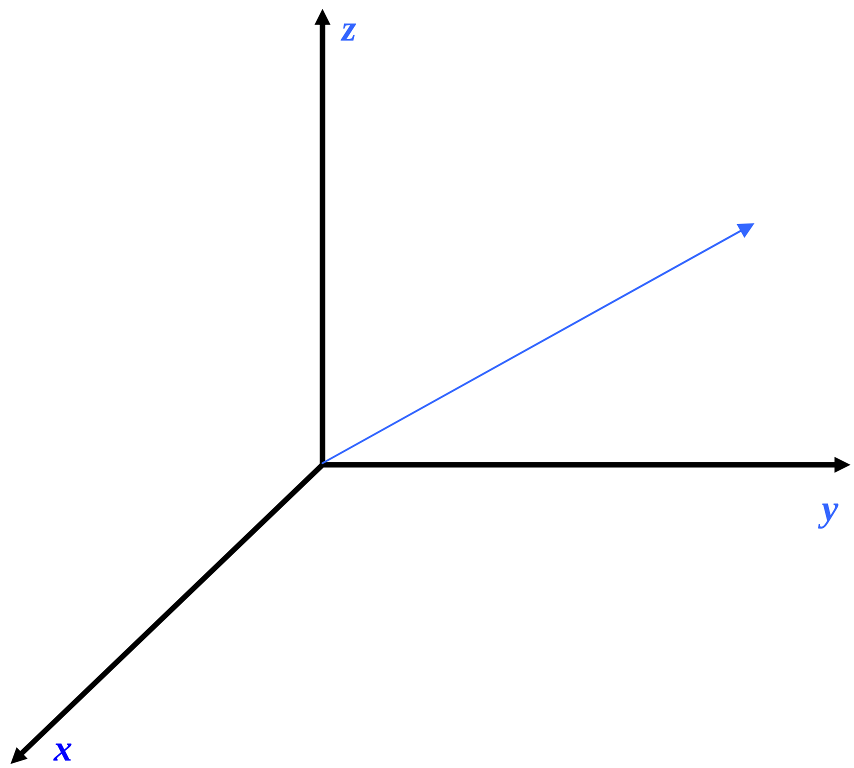
Vectors in 3D Space

Vectors is 3D space are represented by ordered triples $\boldsymbol{v} = < v_1, v_2, v_3 >$. A vector $\boldsymbol{v}$ is standard position has its initial point at the origin (0,0,0) with terminal point given by the ordered triple (v_1, v_2, v_3).

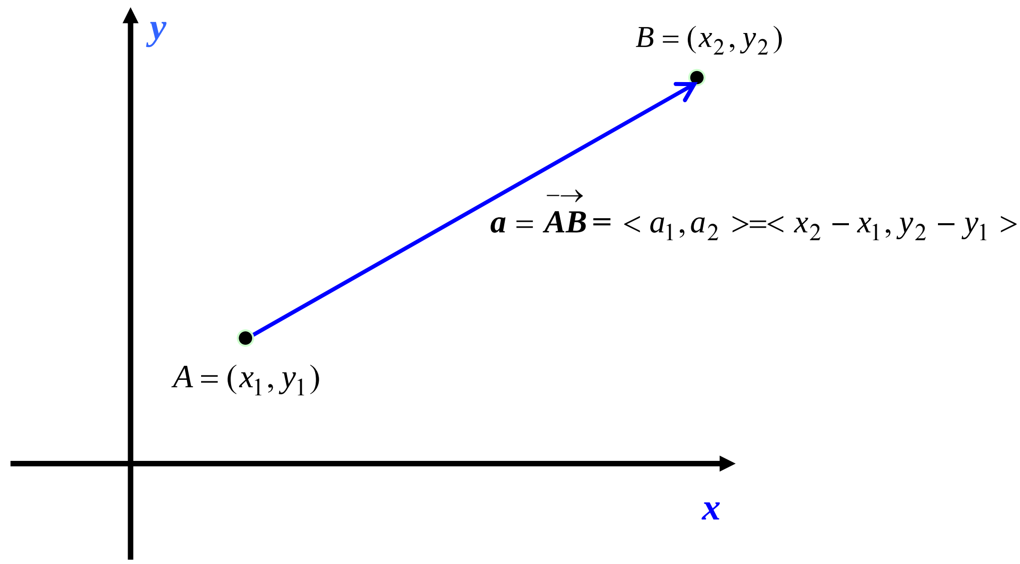


Example 2: Write in component form and sketch the vector in standard position with terminal point (-3, 4, 2).

Solution:

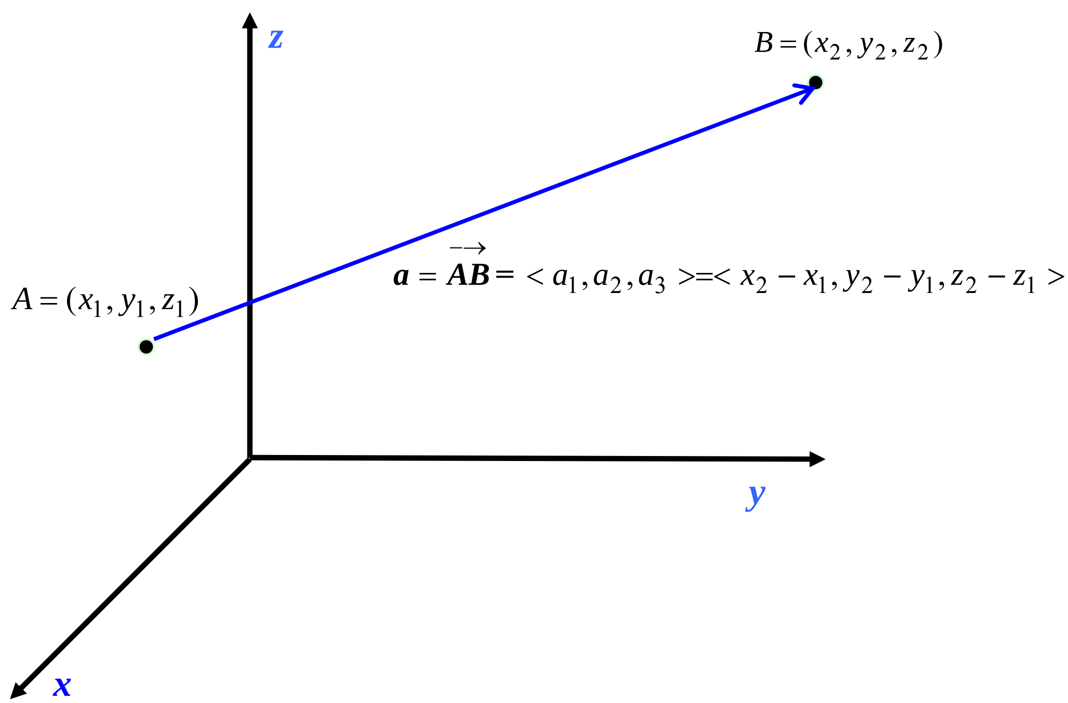
<u>Some Facts about Vectors</u>

1. The zero vector is given by $\mathbf{0} = < 0, 0 >$ in 2D and $\mathbf{0} = < 0, 0, 0 >$ in 3D.
2. Given the points $A = (x_1, y_1)$ and $B = (x_2, y_2)$ in 2D not at the origin.



The component for the vector $\boldsymbol{a}$ is given by $\boldsymbol{a} = \overrightarrow{\boldsymbol{AB}} = < a_1, a_2 >=< x_2 - x_1, y_2 - y_1 >$.

Given the points $A = (x_1, y_1, z_1)$ and $B = (x_2, y_2, z_2)$ in 3D not at the origin.



The component for the vector $\boldsymbol{a}$ is given by $\boldsymbol{a} = \overrightarrow{\boldsymbol{AB}} = < a_1, a_2, a_3 >=< x_2 - x_1, y_2 - y_1, z_2 - z_1 >$.

3. The length (magnitude) of the 2D vector $\boldsymbol{a} = < a_1, a_2 >$ is given by

$$|\,\boldsymbol{a}\,| = \sqrt{a_1^2 + a_2^2}$$

   The length (magnitude) of the 3D vector $\boldsymbol{a} = < a_1, a_2, a_3 >$ is given by

$$|\,\boldsymbol{a}\,| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

4. If $|\,\boldsymbol{a}\,| = 1$, then the vector $\boldsymbol{a}$ is called a *unit vector*.
5. $|\,\boldsymbol{a}\,| = \mathbf{0}$ if and only if $\boldsymbol{a} = \mathbf{0}$.

Example 3: Given the points *A*(3, -5) and *B*(4,7).

a. Find a vector $\boldsymbol{a}$ with representation given by the directed line segment $\overrightarrow{\boldsymbol{AB}}$.
b. Find the length $|\,\boldsymbol{a}\,|$ of the vector $\boldsymbol{a}$.
c. Draw $\overrightarrow{\boldsymbol{AB}}$ and the equivalent representation starting at the origin.

Solution:

Example 4: Given the points $A(2, -1, -2)$and $B(-4, 3, 7)$..

a. Find a vector $\boldsymbol{a}$ with representation given by the directed line segment $\overrightarrow{\boldsymbol{AB}}$.

b. Find the length $|\, \boldsymbol{a} \,|$ of the vector $\boldsymbol{a}$.

c. Draw $\overrightarrow{\boldsymbol{AB}}$ and the equivalent representation starting at the origin.

Solution:

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Facts and Operations With Vectors 2D

Given the vectors ***a*** = $< a_1, a_2 >$ and ***b*** = $< b_1, b_2 >$, k be a scalar. Then the following operations hold.

1. ***a*** + ***b*** = $< a_1, a_2 > + < b_1, b_2 > = < a_1 + b_1, a_2 + b_2 >$. (Vector Addition)
   a - ***b*** = $< a_1, a_2 > - < b_1, b_2 > = < a_1 - b_1, a_2 - b_2 >$. (Vector Subtraction)
2. k ***a*** = $k < a_1, a_2 > = < ka_1, ka_2 >$. (Scalar-Vector Multiplication)
3. Two vectors are equal if and only if their components are equal, that is, ***a*** = ***b*** if and only if $a_1 = b_1$ and $a_2 = b_2$.

Facts and Operations With Vectors 3D

Given the vectors ***a*** = $< a_1, a_2, a_3 >$ and ***b*** = $< b_1, b_2, b_3 >$, k be a scalar. Then the following operations hold.

1. ***a*** + ***b*** = $< a_1, a_2, a_3 > + < b_1, b_2, b_3 > = < a_1 + b_1, a_2 + b_2, a_3 + b_3 >$. (Vector Addition)
   a - ***b*** = $< a_1, a_2, a_3 > - < b_1, b_2, b_3 > = < a_1 - b_1, a_2 - b_2, a_3 - b_3 >$. (Vector Subtraction)
2. k ***a*** = $k < a_1, a_2, a_3 > = < ka_1, ka_2, ka_3 >$. (Scalar-Vector Multiplication)
3. Two vectors are equal if and only if their components are equal, that is, ***a*** = ***b*** if and only if $a_1 = b_1$, $a_2 = b_2$, and $a_3 = b_3$.

Example 5: Given the vectors $\boldsymbol{a} = <3,1>$ and $\boldsymbol{b} = <1,2>$, find

a. $\boldsymbol{a} + \boldsymbol{b}$

b. $2\,\boldsymbol{b}$

c. $3\,\boldsymbol{a} - 2\,\boldsymbol{b}$

d. $|\,3\,\boldsymbol{a} - 2\,\boldsymbol{b}\,|$

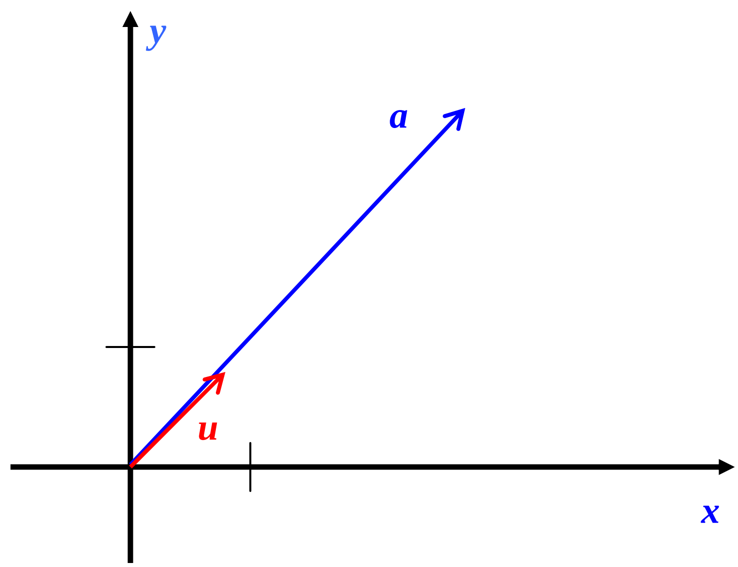
Solution:

Unit Vector in the Same Direction of the Vector *a*

Given a non-zero vector $\boldsymbol{a}$, a <u>unit</u> <u>vector</u> $\boldsymbol{u}$ (vector of length one) in the same direction as the vector $\boldsymbol{a}$ can be constructed by multiplying $\boldsymbol{a}$ by the scalar quantity $\frac{1}{|\boldsymbol{a}|}$, that is, forming

$$\boldsymbol{u} = \frac{1}{|\boldsymbol{a}|}\boldsymbol{a} = \frac{\boldsymbol{a}}{|\boldsymbol{a}|}$$

Multiplying the vector $|\boldsymbol{a}|$ by $\frac{1}{|\boldsymbol{a}|}$ to get the unit vector $\boldsymbol{u}$ is called *normalization*.



Example 6: Given the vector ***a*** = < -4, 5, 3>.

a. Find a unit vector in the same direction as ***a*** and verify that the result is indeed a unit vector.
b. Find a vector that has the same direction as ***a*** but has length 10.

Solution: **Part a)** To compute the unit vector ***u*** in the same direction of ***a*** = < -4, 5, 3>, we first need to find the length of ***a*** which is given by

$$|\,\boldsymbol{a}\,| = \sqrt{(-4)^2 + (5)^2 + (3)^2} = \sqrt{16+25+9} = \sqrt{50} = \sqrt{25 \cdot 2} = 5\sqrt{2}$$

Then

$$\boxed{\boldsymbol{u} = \frac{1}{|\,\boldsymbol{a}\,|}\boldsymbol{a} = \frac{1}{5\sqrt{2}} < -4, 5, 3 > = < -\frac{4}{5\sqrt{2}}, \frac{5}{5\sqrt{2}}, \frac{3}{5\sqrt{2}} > = < -\frac{4}{5\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{3}{5\sqrt{2}} > .}$$

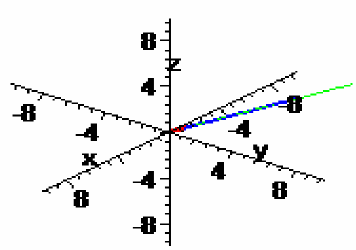
For ***u*** to be a unit vector, we must show that | ***u*** | = 1. Computing the length of | ***u*** | we obtain

$$|\,\boldsymbol{u}\,| = \sqrt{\left(-\frac{4}{5\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{3}{5\sqrt{2}}\right)^2} = \sqrt{\frac{16}{25 \cdot 2} + \frac{1}{2} + \frac{9}{25 \cdot 2}} = \sqrt{\frac{16}{50} + \frac{25}{50} + \frac{9}{50}} = \sqrt{\frac{50}{50}} = \sqrt{1} = 1$$

Part b) Since the unit vector ***u*** found in part a has length 1 is in the same direction of ***a***, multiplying the unit vector ***u*** by 10 will give a vector, which we will call ***b***, with a length of 10, in the same direction of ***a***. Thus,

$$\boldsymbol{b} = 10\boldsymbol{u} = 10 < -\frac{4}{5\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{3}{5\sqrt{2}} > = < -\frac{40}{5\sqrt{2}}, \frac{10}{\sqrt{2}}, \frac{30}{5\sqrt{2}} > = < -\frac{8}{\sqrt{2}}, \frac{10}{\sqrt{2}}, \frac{6}{\sqrt{2}} >$$

The following graph shows the 3 vectors on the same graph, where you can indeed see they are all pointing in the same direction (the unit vector ***u*** is in red, the given vector ***a*** in blue, and the vector ***b*** in green.



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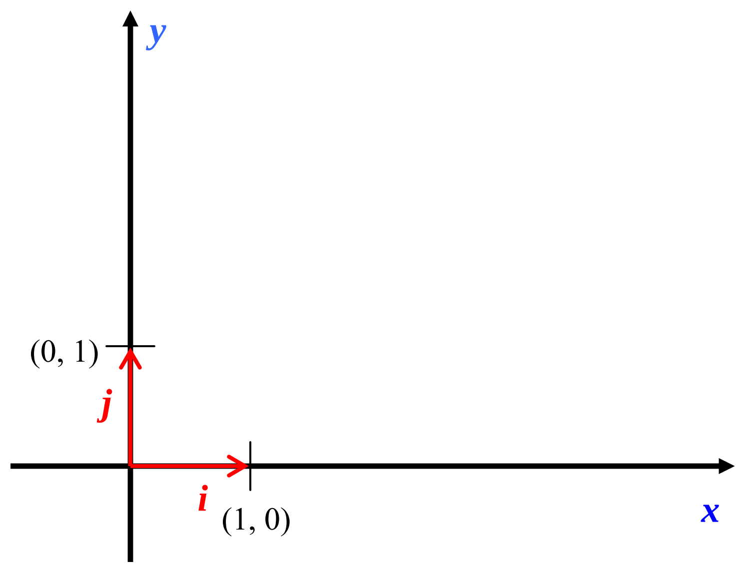
Standard Unit Vectors

In 2D, the unit vectors < 1, 0 > and < 0, 1 > are the *standard unit vectors*. We denote these vectors as ***i*** **=** < 1, 0 > and ***j*** **=** < 0, 1 >. The following represents their graph in the *x*-*y* plane.



Any vector in component form can be written as a linear combination of the standard unit vectors ***i*** and ***j***. That is, the vector ***a*** **=** $< a_1, a_2 >$ in component for can be written

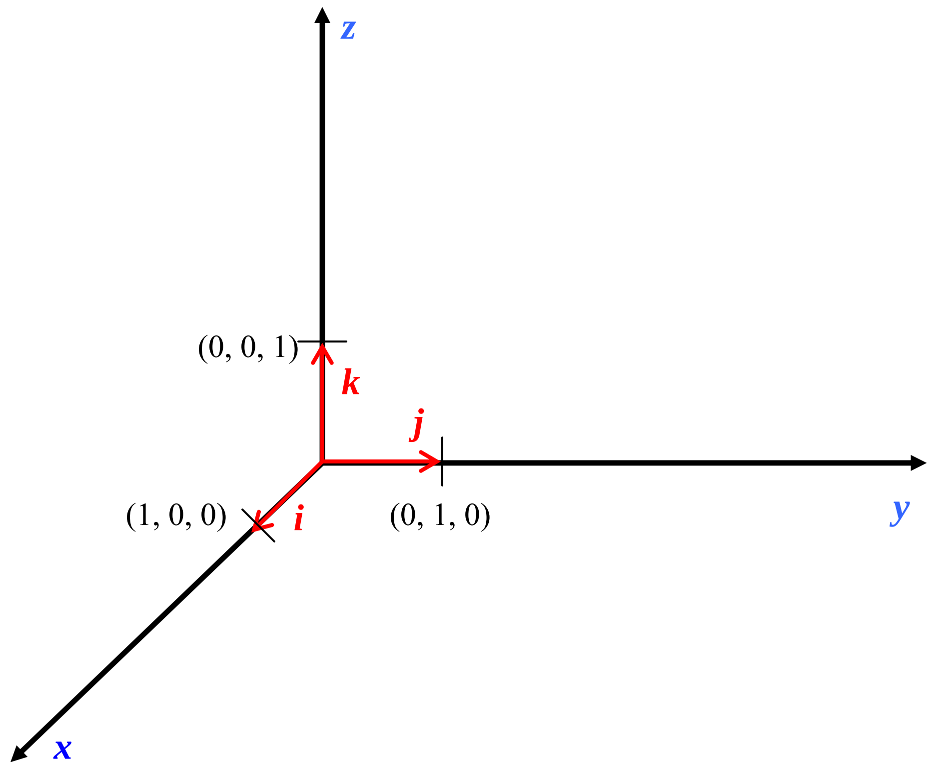
$$\boldsymbol{a} = < a_1, a_2 > = < a_1, 0 > + < 0, a_2 > = a_1 < 1,0 > + a_2 < 0,1 > \; = \; a_1 \, \boldsymbol{i} + \; a_2 \, \boldsymbol{j}$$

in standard unit vector form. For example, the vector < 2, -4 > in component form can be written as

$$< 2,-4 > = 2\boldsymbol{i} - 4\boldsymbol{j}$$

in standard unit vector form.

In 3D, the standard unit vectors are $\boldsymbol{i} = \ < 1, 0, 0 >$, $\boldsymbol{j} = < 0, 1, 0 >$, and $\boldsymbol{k} = \ < 0, 0, 1 >$.



Any vector in component form can be written as a linear combination of the standard unit vectors ***i*** and ***j*** and ***k***. That is, That is, the vector $\boldsymbol{a} = < a_1, a_2, a_3 >$ in component for can be written

$$\boldsymbol{a} = < a_1, a_2, a_3 > = < a_1, 0, 0 > + < 0, a_2, 0 > + < 0, 0, a_3 > = a_1 < 1,0,0 > + a_2 < 0,1,0 > + a_3 < 0,0,1 >$$
$$= a_1 \, \boldsymbol{i} + a_2 \, \boldsymbol{j} + a_3 \, \boldsymbol{k}$$

in standard unit vector from. For example the vector the vector < 2, -4, 5 > in component form can be written as

$$< 2, -4, 5 > = 2\boldsymbol{i} - 4\boldsymbol{j} + 5\boldsymbol{k}$$

in standard unit vector form.

Example 7: Given the vectors $\boldsymbol{a} = \boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k}$, $\boldsymbol{b} = 2\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$, and $\boldsymbol{c} = 4\boldsymbol{i} - 4\boldsymbol{k}$, find

a. $\boldsymbol{a} - \boldsymbol{b}$

b. $|\, \boldsymbol{a} - \boldsymbol{b} \,|$

c. $|\, 5\boldsymbol{a} - 3\boldsymbol{b} - \frac{1}{2}\boldsymbol{c} \,|$

Solution: