Section 9.3: The Dot Product

Practice HW from Stewart Textbook (not to hand in)
p. 655 # 3-8, 11, 13-15, 17, 23-26

Dot Product of Two Vectors

The dot product of two vectors gives a scalar that is computed in the following manner.

In 2D, if $\boldsymbol{a} = < a_1, a_2 >$ and $\boldsymbol{b} = < b_1, b_2 >$, then

Dot product = $\boldsymbol{a} \cdot \boldsymbol{b} = a_1 b_1 + a_2 b_2$

In 3D, if $\boldsymbol{a} = < a_1, a_2, a_3 >$ and $\boldsymbol{b} = < b_1, b_2, b_3 >$, then

Dot product = $\boldsymbol{a} \cdot \boldsymbol{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$

Properties of the Dot Product p, 654
Let $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ be vectors, k be a scalar.
1. $\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{a}$
2. $\boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \cdot \boldsymbol{c}$
3. $0 \cdot \boldsymbol{a} = 0$
4. $k(\boldsymbol{a} \cdot \boldsymbol{b}) = (k\ \boldsymbol{a}) \cdot \boldsymbol{b} = \boldsymbol{a} \cdot (k\ \boldsymbol{b})$
5. $\boldsymbol{a} \cdot \boldsymbol{a} = |\ \boldsymbol{a}\ |^2$

Example 1: Given $\boldsymbol{a} = 2\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k}$ and $\boldsymbol{b} = \boldsymbol{i} - 3\boldsymbol{j} + 2\boldsymbol{k}$, find

a. $\boldsymbol{a} \cdot \boldsymbol{b}$

b. $\boldsymbol{b} \cdot \boldsymbol{a}$

c. $\boldsymbol{a} \cdot \boldsymbol{a}$

d. $|\boldsymbol{a}|^2$

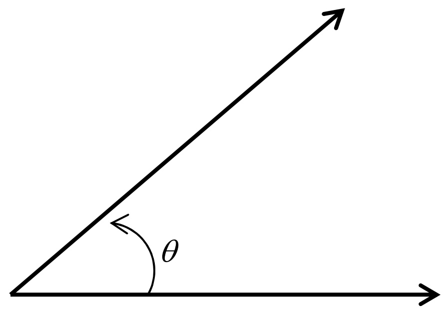
e. $(\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{b}$

f. $\boldsymbol{a} \cdot (2\boldsymbol{v})$

Solution:

Angle Between Two Vectors

Given two vectors ***a*** and ***b*** separated by an angle θ, $0 \le \theta \le \pi$.



Then

$$\cos\theta = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\,\boldsymbol{a}\,|\,|\,\boldsymbol{b}\,|}$$

Then we can write the dot product as

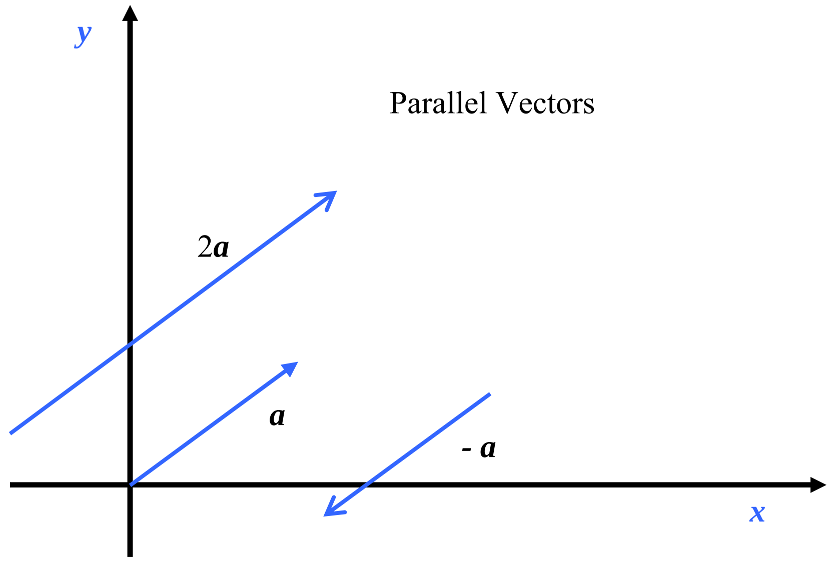
$$\boldsymbol{a} \cdot \boldsymbol{b} = |\,\boldsymbol{a}\,|\,|\,\boldsymbol{b}\,|\cos\theta$$

Example 2: Find the angle θ between the given vectors $\boldsymbol{a} = < 3, 1 >$ and $\boldsymbol{b} = < 2, -1 >$.

Solution:

Parallel Vectors

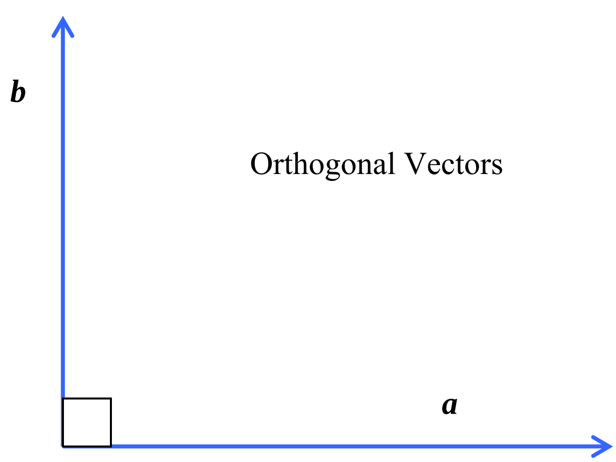
Two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel of there is a scalar k where $\boldsymbol{a} = k\,\boldsymbol{b}$.



Orthogonal Vectors

Two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are orthogonal (intersect at a 90^0 angle) of

$$\boldsymbol{a} \cdot \boldsymbol{b} = 0$$



Note: If two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are orthogonal, they intersect at the angle $\theta = \dfrac{\pi}{2}$ and

$$\boldsymbol{a} \cdot \boldsymbol{b} = |\,\boldsymbol{a}\,|\;|\,\boldsymbol{b}\,|\cos(\frac{\pi}{2}) = |\,\boldsymbol{a}\,|\;|\,\boldsymbol{b}\,|(\mathbf{0}) = \mathbf{0}$$

Example 3: Determine whether the two vectors ***a*** and ***b*** are orthogonal, parallel, or neither.

a. $\boldsymbol{a} =< 2, 18 >$, $\boldsymbol{b} =< \frac{3}{2}, -\frac{1}{6} >$

b. $\boldsymbol{a} = -4\boldsymbol{i} - 5\boldsymbol{j} + 6\boldsymbol{k}$, $\boldsymbol{b} = 8\boldsymbol{i} + 10\boldsymbol{j} - 12\boldsymbol{k}$

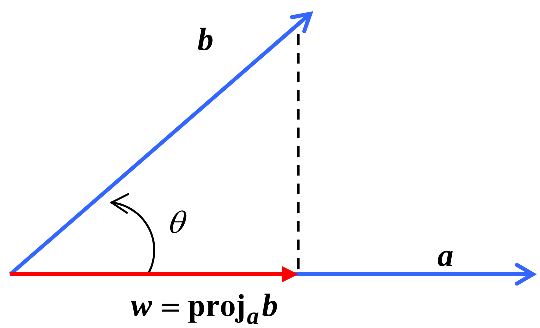
c. $\boldsymbol{a} = -4\boldsymbol{i} - 5\boldsymbol{j} + 6\boldsymbol{k}$, $\boldsymbol{b} = 5\boldsymbol{j} - 6\boldsymbol{k}$

Solution:

$\blacksquare$

Projections

Suppose we are given the vectors ***a*** and ***b*** in the following diagram



The vector in red $\boldsymbol{w} = \mathbf{proj}_{\boldsymbol{a}}\boldsymbol{b}$ is called the *vector projection* of the vector ***b*** onto the vector ***a***. Since ***w*** is a smaller vector in length the vector ***a***, it is "parallel" to ***a*** and hence is a scalar multiple of ***a***. Thus, we can write ***w*** = *k* ***a***, The scalar *k* is know as the *scalar projection* of vector ***b*** onto the vector ***a*** (also known as the *component* of ***b*** along ***a***). We assign the scalar *k* the notation

$$k = \boldsymbol{comp}_{\boldsymbol{a}}\boldsymbol{b}$$

Our goal first is to find *k*. From the definition of a right triangle,

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{|\boldsymbol{w}|}{|\boldsymbol{b}|} = \frac{|\boldsymbol{ka}|}{|\boldsymbol{b}|} = \frac{\boldsymbol{k}\,|\boldsymbol{a}|}{|\boldsymbol{b}|}$$

Also, the definition of the dot product says that

$$\cos\theta = \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}|\,|\boldsymbol{b}|}$$

Hence, we can say that

$$\frac{\boldsymbol{k}\,|\boldsymbol{a}|}{|\boldsymbol{b}|} = \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}|\,|\boldsymbol{b}|}$$

Solving for *k* gives

$$\boldsymbol{k} = \frac{|\boldsymbol{b}|}{|\boldsymbol{a}|}\frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}|\,|\boldsymbol{b}|} = \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}|^2}$$

To get the vector projection, we compute the vector $\boldsymbol{w}$. This gives the following result.

$$\boldsymbol{w} = k\boldsymbol{a} = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\,\boldsymbol{a}\,|^2}\boldsymbol{a}$$

Summarizing, we obtain the following results.

Scalar and Vector Projection

Scalar Projection of $\boldsymbol{b}$ onto $\boldsymbol{a}$: $\quad \mathbf{comp}_{\boldsymbol{a}}\boldsymbol{b} = \dfrac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\,\boldsymbol{a}\,|^2}$

Vector Projection of $\boldsymbol{b}$ onto $\boldsymbol{a}$: $\quad \mathbf{proj}_{\boldsymbol{a}}\boldsymbol{b} = (\mathbf{comp}_{\boldsymbol{a}}\boldsymbol{b}\,)\,\boldsymbol{a} = \dfrac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\,\boldsymbol{a}\,|^2}\;\boldsymbol{a}$

Example 4: Find the scalar and vector projections of $\boldsymbol{b}$ onto $\boldsymbol{a}$ if $\boldsymbol{a} =< 0,2,3 >$ and $< -2,1,1 >$

Solution: The scalar projection of of $\boldsymbol{b}$ onto $\boldsymbol{a}$ is given by the formula

$$\mathbf{comp}_{\boldsymbol{a}}\boldsymbol{b} = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\,\boldsymbol{a}\,|^2}$$

We see that

$$\boldsymbol{a} \cdot \boldsymbol{b} = (0)(-2) + (2)(1) + (3)(1) = 0 + 2 + 3 = 5$$

and that

$$|\,\boldsymbol{a}\,|^2 = \left(\sqrt{(0)^2 + (2)^2 + (3)^2}\right)^2 = \left(\sqrt{13}\right)^2 = 13\,.$$

Thus the scalar projection is

$$\text{Scalar projection of } \boldsymbol{b} \text{ onto } \boldsymbol{a} \; = \; \mathbf{comp}_{\boldsymbol{a}}\boldsymbol{b} = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\,\boldsymbol{a}\,|^2} = \frac{5}{13}\,.$$

Hence, the vector projection is

$$\text{Vector projection of } \boldsymbol{b} \text{ onto } \boldsymbol{a} \; = \; (\mathbf{comp}_{\boldsymbol{a}}\boldsymbol{b})\,\boldsymbol{a} = \frac{5}{13} < 0,2,3 > = < 0, \frac{10}{13}, \frac{15}{13} > .$$