Section 9.4: The Cross Product

Practice HW from Stewart Textbook (not to hand in)
p. 664 # 1, 7-17

Cross Product of Two Vectors

The cross product of two vectors produces a vector (unlike the dot product which produces s scalar) that has important properties. Before defining the cross product, we first give a method for computing a 2×2 determinant.

Definition: The *determinant* of a 2×2 matrix, denoted by $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$, is defined to be the scalar

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Example 1: Compute $\begin{vmatrix} 2 & -3 \\ 4 & 5 \end{vmatrix}$

Solution:

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We next define the cross product of two vectors.

Definition: If $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k}$ and $\boldsymbol{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j} + b_3\boldsymbol{k}$ be vectors in 3D space. The cross product is the vector

$$\boldsymbol{a}\times\boldsymbol{b} = (a_2b_3 - a_3b_2)\,\boldsymbol{i} + (a_1b_3 - a_3b_1)\boldsymbol{j} + (a_1b_2 - a_2b_1)\,\boldsymbol{k}$$

To calculate the cross product more easily without having to remember the formula, we using the following "determinant" form.

$$\boldsymbol{a}\times\boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \boldsymbol{a}_1 & \boldsymbol{a}_2 & \boldsymbol{a}_3 \\ \boldsymbol{b}_1 & \boldsymbol{b}_2 & \boldsymbol{b}_3 \end{vmatrix} \begin{array}{l} \leftarrow \text{Standard unit vectors in row 1} \\ \leftarrow \text{Components of left vector } \boldsymbol{a} \text{ in row 2} \\ \leftarrow \text{Components of left vector } \boldsymbol{b} \text{ in row 2} \end{array}$$

We calculate the 3×3 determinant as follows: (note the alternation in sign)

$$\boldsymbol{a}\times\boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \boldsymbol{a}_1 & \boldsymbol{a}_2 & \boldsymbol{a}_3 \\ \boldsymbol{b}_1 & \boldsymbol{b}_2 & \boldsymbol{b}_3 \end{vmatrix} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \boldsymbol{a}_1 & \boldsymbol{a}_2 & \boldsymbol{a}_3 \\ \boldsymbol{b}_1 & \boldsymbol{b}_2 & \boldsymbol{b}_3 \end{vmatrix} - \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \boldsymbol{a}_1 & \boldsymbol{a}_2 & \boldsymbol{a}_3 \\ \boldsymbol{b}_1 & \boldsymbol{b}_2 & \boldsymbol{b}_3 \end{vmatrix} + \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \boldsymbol{a}_1 & \boldsymbol{a}_2 & \boldsymbol{a}_3 \\ \boldsymbol{b}_1 & \boldsymbol{b}_2 & \boldsymbol{b}_3 \end{vmatrix}$$

$$= \boldsymbol{i}\begin{vmatrix} \boldsymbol{a}_2 & \boldsymbol{a}_3 \\ \boldsymbol{b}_2 & \boldsymbol{b}_3 \end{vmatrix} - \boldsymbol{j}\begin{vmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_3 \\ \boldsymbol{b}_1 & \boldsymbol{b}_3 \end{vmatrix} + \boldsymbol{k}\begin{vmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 \\ \boldsymbol{b}_1 & \boldsymbol{b}_2 \end{vmatrix}$$

$$= \boldsymbol{i}(\boldsymbol{a}_2\boldsymbol{b}_3 - \boldsymbol{a}_3\boldsymbol{b}_2) - \boldsymbol{j}(\boldsymbol{a}_1\boldsymbol{b}_3 - \boldsymbol{a}_3\boldsymbol{b}_1) + \boldsymbol{k}(\boldsymbol{a}_1\boldsymbol{b}_2 - \boldsymbol{a}_2\boldsymbol{b}_1)$$

Example 2: Given the vectors $\boldsymbol{a} = \boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k}$ and $\boldsymbol{b} = -2\boldsymbol{i} + 3\boldsymbol{j} - \boldsymbol{k}$. Find

a. $\boldsymbol{a} \times \boldsymbol{b}$

b. $\boldsymbol{b} \times \boldsymbol{a}$

c. $\boldsymbol{a} \times \boldsymbol{a}$

[redacted]

Properties of the Cross Product

Let ***a***, ***b***, and ***c*** be vectors, *k* be a scalar.

1. $\boldsymbol{a} \times \boldsymbol{b} = -(\boldsymbol{b} \times \boldsymbol{a})$ Note! $\boldsymbol{a} \times \boldsymbol{b} \neq (\boldsymbol{b} \times \boldsymbol{a})$
2. $\boldsymbol{a} \times (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \times \boldsymbol{b} + \boldsymbol{a} \times \boldsymbol{c}$
3. $k(\boldsymbol{a} \times \boldsymbol{b}) = (k\ \boldsymbol{a}) \times \boldsymbol{b} = \boldsymbol{a} \times (k\ \boldsymbol{b})$
4. $0 \times \boldsymbol{a} = \boldsymbol{a} \times 0 = 0$
5. $\boldsymbol{a} \times \boldsymbol{a} = 0$

Geometric Properties of the Cross Product

Let ***a*** and ***b*** be vectors

1. $\boldsymbol{a} \times \boldsymbol{b}$ is orthogonal to both ***a*** and ***b***.

Example 3: Given the vectors $\boldsymbol{a} = \boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k}$ and $\boldsymbol{b} = -2\boldsymbol{i} + 3\boldsymbol{j} - \boldsymbol{k}$, show that the cross product $\boldsymbol{a} \times \boldsymbol{b}$ is orthogonal to both ***a*** and ***b***.

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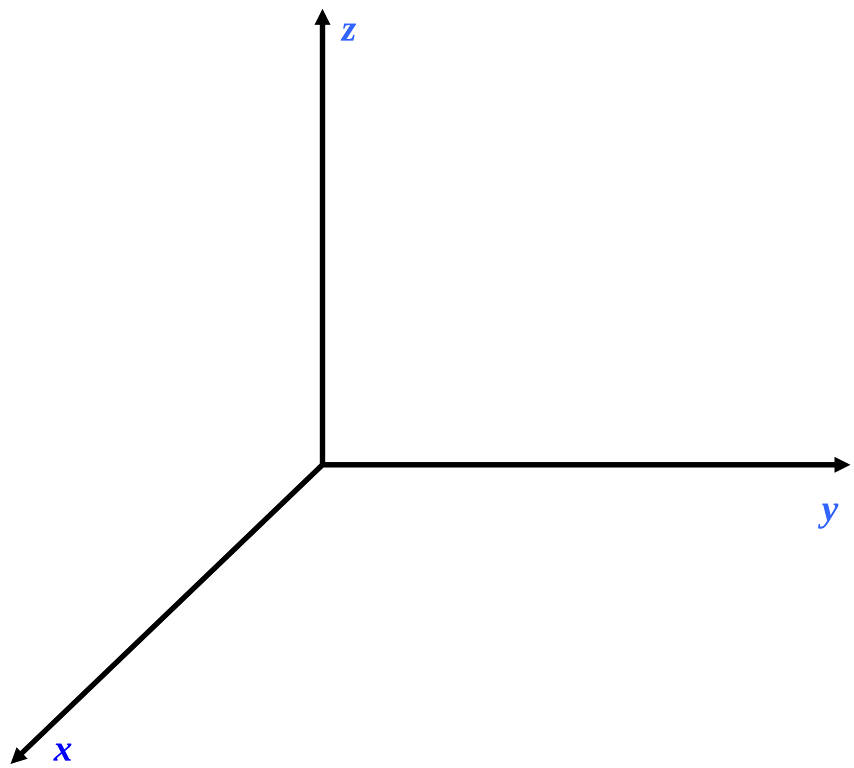
Note: $\boldsymbol{i} \times \boldsymbol{j} = \boldsymbol{k}$, $\boldsymbol{i} \times \boldsymbol{k} = -\boldsymbol{j}$, $\boldsymbol{j} \times \boldsymbol{k} = \boldsymbol{i}$



2. $|\boldsymbol{a} \times \boldsymbol{b}| = |\boldsymbol{a}|\,|\boldsymbol{b}|\,\sin\theta$
3. $\boldsymbol{a} \times \boldsymbol{b} = 0$ if and only $\boldsymbol{a} = k\,\boldsymbol{b}$, that is, if the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel.
4. $|\boldsymbol{a} \times \boldsymbol{b}|$ gives the area having the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ as its adjacent sides.

Example 4: Given the points $P(0, -2, 0)$, $Q(-1, 3, 4)$, and $R(3,0,6)$.
a. Find a vector orthogonal to the plane through these points.
b. Find the area of the parallelogram with the vectors $\overrightarrow{\boldsymbol{PQ}}$ and $\overrightarrow{\boldsymbol{PR}}$ as its adjacent sides.
c. Find area of the triangle PQR.

Solution: Part a) The plane containing the given points will have the vectors $\overrightarrow{\boldsymbol{PQ}}$ and $\overrightarrow{\boldsymbol{PR}}$ as its adjacent sides. We first compute these vectors as follows:

The vector connecting $P(0, -2, 0)$ and $Q(-1, 3, 4)$ is $\overrightarrow{\boldsymbol{PQ}} = < -1-0, 3--2, 4-0 > = < -1, 5, 4 >$.

The vector connecting $P(0, -2, 0)$ and $R(3,0,6)$ is $\overrightarrow{\boldsymbol{PR}} = < 3-0, 0--2, 6-0 > = < 3, 2, 6 >$.

The vector orthogonal to the plane will be the vectors orthogonal to $\overrightarrow{\boldsymbol{PQ}}$ and $\overrightarrow{\boldsymbol{PR}}$, which is precisely the cross product $\overrightarrow{\boldsymbol{PQ}} \times \overrightarrow{\boldsymbol{PR}}$. Thus, we have

$$\overrightarrow{\boldsymbol{PQ}} \times \overrightarrow{\boldsymbol{PR}} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -1 & 5 & 4 \\ 3 & 2 & 6 \end{vmatrix}$$

$$= \boldsymbol{i}\begin{vmatrix} 5 & 4 \\ 2 & 6 \end{vmatrix} - \boldsymbol{j}\begin{vmatrix} -1 & 4 \\ 3 & 6 \end{vmatrix} + \boldsymbol{k}\begin{vmatrix} -1 & 5 \\ 3 & 2 \end{vmatrix}$$

$$= \boldsymbol{i}(5\cdot 6 - 4\cdot 2) - \boldsymbol{j}(-1\cdot 6 - 4\cdot 3) + \boldsymbol{k}(-1\cdot 2 - 5\cdot 3)$$

$$\boxed{= 22\boldsymbol{i} + 18\boldsymbol{j} - 17\boldsymbol{k}}$$

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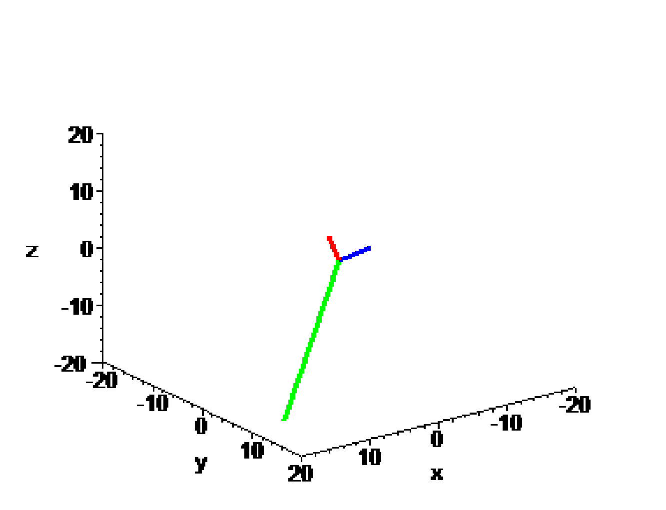
The following displays a graph of the vectors $\overrightarrow{\boldsymbol{PQ}}$ (in blue), $\overrightarrow{\boldsymbol{PR}}$ (in red), and their cross product $\overrightarrow{\boldsymbol{PQ}} \times \overrightarrow{\boldsymbol{PR}}$ (in green).



Part b) The area of the parallelogram with the vectors $\overrightarrow{\boldsymbol{PQ}}$ and $\overrightarrow{\boldsymbol{PR}}$ as its adjacent sides is precisely the length of the cross product of these two vectors that we calculated in part a. Using the fact that $\overrightarrow{\boldsymbol{PQ}} \times \overrightarrow{\boldsymbol{PR}} = 22\boldsymbol{i} + 18\boldsymbol{j} - 17\boldsymbol{k}$, we have that

$$\text{Area of Parallelogram} = \left| \overrightarrow{\boldsymbol{PQ}} \times \overrightarrow{\boldsymbol{PR}} \right| = \sqrt{(22)^2 + (18)^2 + (-17)^2} = \sqrt{484 + 324 + 289} = \sqrt{1097} \approx 33.1 \text{ square units}$$

Part c.) The area of the triangle *PQR* represents exactly one-half of the area of the parallelogram with the vectors $\overrightarrow{\boldsymbol{PQ}}$ and $\overrightarrow{\boldsymbol{PR}}$ as its adjacent sides that we found in part b. Hence, we have

$$\text{Area of Triangle PQR} = \frac{1}{2}(\text{Area of the Parallelogram}) = \frac{1}{2}\left| \overrightarrow{\boldsymbol{PQ}} \times \overrightarrow{\boldsymbol{PR}} \right| = \frac{1}{2}\sqrt{1097} \approx 16.6 \text{ square units}$$

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