Section 10.2: Derivatives and Integrals of Vector Functions

Practice HW from Stewart Textbook (not to hand in)
p. 707 # 3-21 odd, 29-35

Differentiation of Vector Functions

Differentiation of vector valued functions are done component wise in the natural way. Thus, for

1. 2D Case: If $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j}$, then $\mathbf{r}'(t) = f'(t)\,\mathbf{i} + g'(t)\,\mathbf{j}$

2. 3D Case: If $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}$, then $\mathbf{r}'(t) = f'(t)\,\mathbf{i} + g'(t)\,\mathbf{j} + h'(t)\,\mathbf{k}$

Example 1: Find the derivative of the vector function

$$\mathbf{r}(t) = < \frac{1}{t},\ 16t,\ \sqrt{t} > .$$

Solution:

Example 2: Find the derivative of the vector function

$$\mathbf{r}(t) = e^{2t}\ \mathbf{i} + \sin^2 t\ \mathbf{j} + \ln(t^2 + 1)\ \mathbf{k}$$

Solution:

Note: Look at properties involving the derivative of vector value functions on p. 705 Theorem 3 of Stewart text.

Tangent Vector to a Vector Valued Function

Recall that the derivative provides the tool for finding the tangent line to a curve. This same idea can be used to find a vector tangent to a curve at a point. We illustrate this idea in the following example.

Example 3: For the vector function $\mathbf{r}(t) = e^t\ \mathbf{i} + e^{-t}\ \mathbf{j}$,

a. Sketch the plane curve with the given vector equation.
b. Find $\mathbf{r}'(t)$
c. Sketch the positive vector $\mathbf{r}(t)$ and the tangent vector $\mathbf{r}'(t)$ at $t = 0$.

Solution:

Example 4: Find the parametric equations for the tangent line to the curve with the parametric equations $x = t^2 - 1, y = t^2 + 1, z = t + 1$ at the point (-1, 1, 1).

Solution:

Note: Sometimes, it is convenient the normalize a vector tangent to a vector valued function. This gives the *unit tangent vector*.

Unit Tangent Vector

Given a vector function **r** on an open interval I, the unit tangent vector **T**(*t*) is given by

$$\boldsymbol{T}(\mathrm{t}) = \frac{\boldsymbol{r}'(\mathrm{t})}{|\boldsymbol{r}'(\mathrm{t})|} = \frac{1}{|\boldsymbol{r}'(\mathrm{t})|}\boldsymbol{r}'(\mathrm{t}) \text{ where } \boldsymbol{r}'(\mathrm{t}) \neq 0$$

Example 5: Find the unit tangent vector **T**(*t*) for $\mathbf{r}(t) = 2\cos t\ \mathbf{i} + \sin 6t\ \mathbf{j} + 2t\ \mathbf{k}$ at $t = \frac{\pi}{6}$.

Solution: The unit tangent vector is given by the formula

$$\boldsymbol{T}(\mathrm{t}) = \frac{\boldsymbol{r}'(\mathrm{t})}{|\boldsymbol{r}'(\mathrm{t})|} = \frac{1}{|\boldsymbol{r}'(\mathrm{t})|}\boldsymbol{r}'(\mathrm{t}).$$

Using $\mathbf{r}(t) = 2\cos \boldsymbol{t}\ \mathbf{i} + \sin 6t\ \mathbf{j} + 2t\ \mathbf{k}$, we see that

$$\boldsymbol{r}'(\mathrm{t}) = -2\sin \mathrm{t}\,\boldsymbol{i} + 6\cos 6\mathrm{t}\,\boldsymbol{j} + 2\boldsymbol{k} = < -2\sin \mathrm{t}, 6\cos 6\mathrm{t}, 2 >$$

and

$$|\boldsymbol{r}'(\mathrm{t})| = \sqrt{(-2\sin \mathrm{t})^2 + (6\cos 6\mathrm{t})^2 + (2)^2} = \sqrt{4\sin^2 \mathrm{t} + 36\cos^2 6\mathrm{t} + 4}$$

Thus,

$$\boldsymbol{T}(\mathrm{t}) = \frac{1}{|\boldsymbol{r}'(\mathrm{t})|}\boldsymbol{r}'(\mathrm{t}) = \frac{1}{\sqrt{4\sin^2 \mathrm{t} + 36\cos^2 6\mathrm{t} + 4}} < -2\sin \mathrm{t}, 6\cos 6\mathrm{t}, 2 >$$

At $t = \frac{\pi}{6}$, we have **(continued on next page)**

$$\boldsymbol{T}(\frac{\pi}{6}) = \frac{1}{\sqrt{4\sin^2(\pi/6) + 36\cos^2(6\pi/6) + 4}} < -\boldsymbol{s}2\boldsymbol{in}(\pi/6)\,, 6\cos(6\pi/6), 2 >$$

$$= \frac{1}{\sqrt{(1/2)^2 + 36(-1)^2 + 4}} < -2(\frac{1}{2})\,, 6(-1), 2 > \quad \text{(Note that } \sin(\pi/6) = 1/2\,,\ \cos(6\pi/6) = \cos\pi = \text{-1)}$$

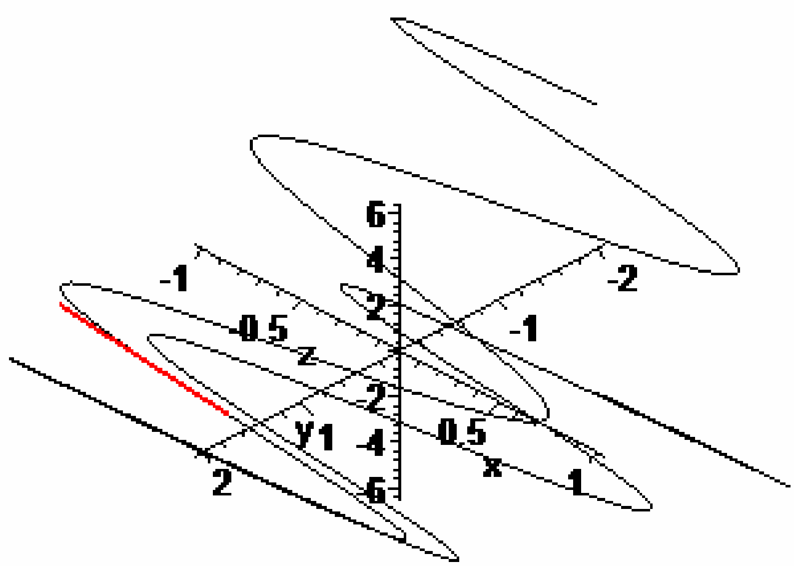
$$= \frac{1}{\sqrt{4(\frac{1}{4}) + 36(1) + 4}} < -1, -6, 2 >$$

$$= \frac{1}{\sqrt{1 + 36 + 4}} < -1, -6, 2 >$$

$$= \frac{1}{\sqrt{41}} < -1\,, -6, 2 >$$

$$= < -\frac{1}{\sqrt{41}}, -\frac{6}{\sqrt{41}}, \frac{2}{\sqrt{41}} >$$

The following graph plots in 3D space the vector function **r**(t) and the corresponding unit tangent vector **T**(t) evaluated at $t = \frac{\pi}{6}$.



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Integrals of Vector Functions

Integrals of Vector Valued Functions are computed component wise.

1. 2D Case: If $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j}$, then $\int \mathbf{r}(t)dt = \int f(t)dt\ \mathbf{i} + \int g(t)dt\ \mathbf{j}$
2. 3D Case: If $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}$, then $\int \mathbf{r}(t)dt = \int f(t)dt\ \mathbf{i} + \int g(t)dt\ \mathbf{j} + \int h(t)dt\ \mathbf{k}$

Example 6: Evaluate the integral $\int (3t^2\,\mathbf{i} + 4t\,\mathbf{j} - 8t^3\,\mathbf{k})dt$

Solution:

$\blacksquare$

Example 7: Evaluate the integral $\int_0^{\pi} (t^2 e^{t^3}\,\mathbf{i} + \sin t\,\mathbf{j} - 2t\,\mathbf{k})dt$

Solution:

$\blacksquare$

Example 8: Find $\mathbf{r}(t)$ if $\mathbf{r}'(t) = 2t\,\mathbf{j} + \sqrt{t}\,\mathbf{k}$ and $\mathbf{r}(0) = \mathbf{i} + \mathbf{j}$.

Solution: Writing $\mathbf{r}'(t) = 2t\,\mathbf{j} + \sqrt{t}\,\mathbf{k} = 2t\,\mathbf{j} + (t)^{\frac{1}{2}}\,\mathbf{k}$, we see that

$$\begin{aligned}
\mathbf{r}(t) &= \int \mathbf{r}'(t)dt \\
&= \int (2t\,\mathbf{i} + t^{\frac{1}{2}}\,\mathbf{k})dt \\
&= 2\frac{t^2}{2}\mathbf{i} + \frac{t^{\frac{3}{2}}}{\frac{3}{2}}\mathbf{k} + \mathbf{C} \\
&= t^2\mathbf{i} + \frac{2}{3}t^{\frac{3}{2}}\,\mathbf{k} + \mathbf{C}
\end{aligned}$$

Thus, $\mathbf{r}(t) = t^2\mathbf{i} + \frac{2}{3}t^{\frac{3}{2}}\,\mathbf{k} + \mathbf{C}$ and we need to use the initial condition $\mathbf{r}(0) = \mathbf{i} + \mathbf{j}$ to find the constant vector $\mathbf{C}$. We see that

$$\mathbf{i} + \mathbf{j} = \mathbf{r}(0) = (0)^2\mathbf{i} + \frac{2}{3}(0)^{\frac{3}{2}}\,\mathbf{k} + \mathbf{C}$$

which gives $\mathbf{C} = \mathbf{i} + \mathbf{j}$. Thus, substituting for $\mathbf{C}$ gives

$$\mathbf{r}(t) = t^2\mathbf{i} + \frac{2}{3}t^{\frac{3}{2}}\,\mathbf{k} + (\mathbf{i} + \mathbf{j}),$$

which, when combining like terms, gives the result.

$$\boxed{\mathbf{r}(t) = (t^2 + \mathbf{1})\mathbf{i} + \mathbf{j} + \frac{2}{3}t^{\frac{3}{2}}\,\mathbf{k}}$$

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