Section 10.4: Motion in Space: Velocity and Acceleration

Practice HW from Stewart Textbook (not to hand in)
p. 725 # 3-17 odd, 21, 23

Velocity and Acceleration

Given a vector function $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}$, we define the following quantities:

Velocity $= \mathbf{v}(t) = \mathbf{r}'(t) = f'(t)\,\mathbf{i} + g'(t)\,\mathbf{j} + h'(t)\,\mathbf{k}$

Speed $= |\,\mathbf{v}(t)\,| = |\,\mathbf{r}'(t)\,| = \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}$

Acceleration $= \mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t) = f''(t)\,\mathbf{i} + g''(t)\,\mathbf{j} + h''(t)\,\mathbf{k}$

Example 1: Find the velocity, acceleration, and speed of a particle given by the position function $\mathbf{r}(t) =< t+1, t^2 >$ at t = 2. Sketch the path of the particle and draw the velocity and acceleration vectors for the specified value of t.

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Solution:

Example 2: Find the velocity, acceleration, and speed of a particle given by the position function $\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 3\sin t\,\mathbf{j}$ at $t = 0$. Sketch the path of the particle and draw the velocity and acceleration vectors for the specified value of t.

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Solution: We first calculate the velocity, speed, and acceleration formulas for an arbitrary value of t. In the process, we substitute and find each of these vectors at $t = 0$.

$$\textit{Velocity} = \mathbf{v}(t) = \mathbf{r}'(t) = -2\sin t\,\mathbf{i} + 3\cos t\,\mathbf{j}$$

$$\begin{matrix}\textit{Velocity at} \\ t=0\end{matrix} = \mathbf{v}(0) = -2\sin 0\,\mathbf{i} + 3\cos 0\,\mathbf{j} = -2(0\_\,\mathbf{i} + 3(1)\mathbf{j} = 3\mathbf{j} = <0,3>$$

$$\textit{Speed} = |\mathbf{v}(t)| = \sqrt{(-2\sin t)^2 + (3\cos t)^2} = \sqrt{4\sin^2 t + 9\cos^2 t}$$

$$\begin{matrix}\textit{Speed at} \\ t=0\end{matrix} = |\mathbf{v}(0)| = \sqrt{4\sin^2 0 + 9\cos^2 0} = \sqrt{4(0)^2 + 9(1)^2} = \sqrt{0+9} = \sqrt{9} = 3$$

$$\textit{Acceleration} = \mathbf{a}(t) = \mathbf{v}'(t) = -2\cos t\,\mathbf{i} - 3\sin t\,\mathbf{j}$$

$$\begin{matrix}\textit{Acceleration at} \\ t=0\end{matrix} = \mathbf{a}(0) = -2\cos 0\,\mathbf{i} - 3\sin 0\,\mathbf{j} = -2(1)\mathbf{i} - 3(0)\mathbf{j} = -2\mathbf{i} = <-2,0>$$

To graph the path of the particle, we take the position $\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 3\sin t\,\mathbf{j}$, write the parametric equations $x = 2\cos t$ and $y = 3\sin t$, and solve for the trigonometric terms to get $\cos t = \frac{x}{2}$ and $\sin t = \frac{y}{3}$. Rewriting the Pythagorean identity $\cos^2 t + \sin^2 t = 1$ as

$$(\cos t)^2 + (\sin t)^2 = 1$$

and substituting $\cos t = \frac{x}{2}$ and $\sin t = \frac{y}{3}$, we obtain

$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$$

or

$$\frac{x^2}{4} + \frac{y^2}{9} = 1$$

This is the equation of an ellipse with center (0, 0). The vertices on the x-axis are (2, 0) and (-2, 0) formed by taking $a = \sqrt{4} = 2$ and moving to the left and right of the center on

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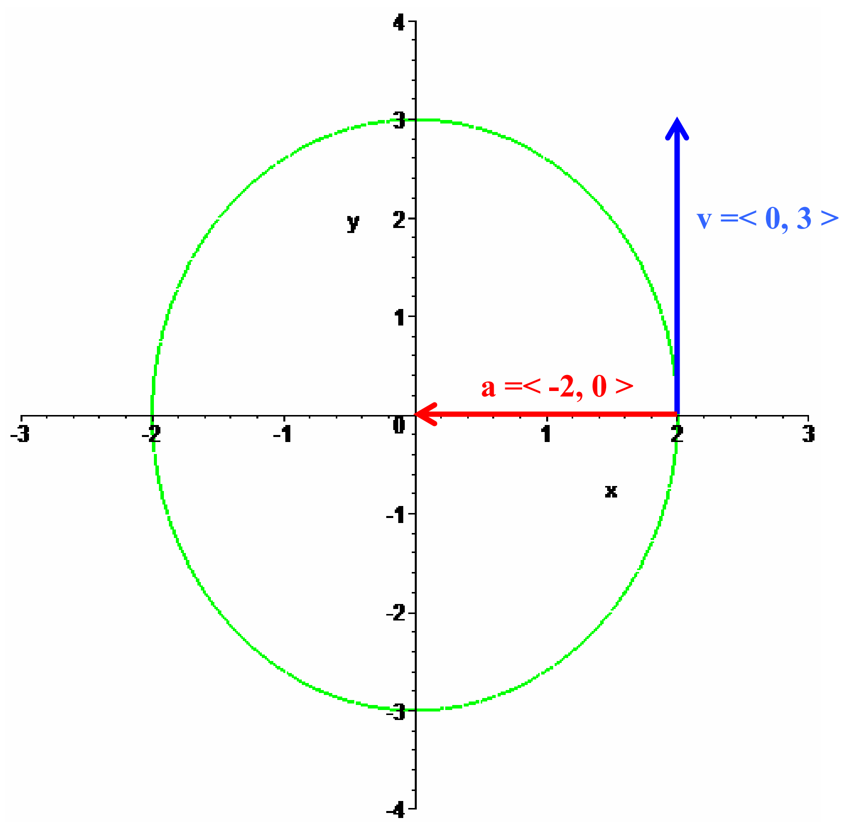
the x-axis. The vertices on the y-axis are (0, 3) and (0, -3) formed by taking $b = \sqrt{9} = 3$ and moving up and down from the center on the y-axis. To find the position of the particle at time $t = 0$, we take the position function $\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 3\sin t\,\mathbf{j}$ and substitute $t = 0$ to obtain

$$\mathbf{r}(0) = 2\cos 0\,\mathbf{i} + 3\sin 0\,\mathbf{j} = 2(1)\,\mathbf{i} + 3(0)\mathbf{j} = 2\,\mathbf{i} =< 2,0 > .$$

The terminal point of this vector, (2, 0), gives the initial point to plot the velocity and acceleration vectors at $t = 0$ we found above, $\mathbf{v}(0) =< 0,3 >$ and $\mathbf{a}(0) =< -2,0 >$. The following shows the result of the graph, with the equation of the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$ plotted in green, the velocity vector $\mathbf{v} = < 0, 3 >$ in blue, and the acceleration vector $\mathbf{a} = < -2,\ 0 >$ in red.



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Example 3: Find the velocity, acceleration, and speed of a particle given by the position function $\mathbf{r}(t) = e^{t^2}\,\mathbf{i} + \sin 4t\,\mathbf{j} + t^2\,\mathbf{k}$

Solution:

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Example 4: Given the acceleration $\mathbf{a}(t) = \mathbf{i} + t\,\mathbf{k}$ with initial velocity of $\mathbf{v}(0) = 5\mathbf{j}$ and initial position of $\mathbf{r}(0) = \mathbf{i}$, find the velocity and position vector functions.

Solution:

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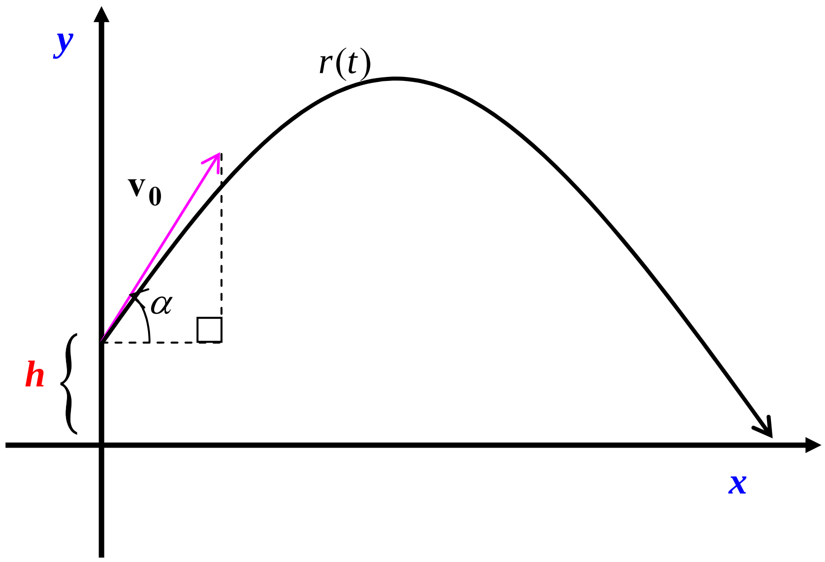
Projectile Motion

Consider the path of a projectile with initial velocity vector $\mathbf{v}_0$ and initial height h launched at an angle α with the ground.



The vector function $\mathbf{r}(t)$ describing the projectile (a derivation of this equation is described on p. 719 of the Stewart text) is given as follows.

Projectile Motion

Neglecting air resistance, the path of a projectile at time t launched from an initial height h with initial speed v_0 and angle of elevation α is described by the vector function

$$\mathbf{r}(t) = (v_0 \cos\alpha)t \ \mathbf{i} + [h + (v_0 \sin\alpha)t - \frac{1}{2}gt^2]\ \mathbf{j}$$

where g is acceleration due to gravity ($g = 9.8\, m/s^2$ in the metric system or $g = 32\ ft/s^2$ in the English system). Note, we the parametric equations of this function can be used to describe the horizontal and vertical position of the projectile. That is, $x = (v_0 \cos\alpha)t$ describes the horizontal position of the projectile and $y = h + (v_0 \sin\alpha)t - \frac{1}{2}gt^2$ describes the vertical position of the projectile.

Example 5: A baseball is hit 4 feet above the ground leaves the bat with an initial speed of 98 ft/sec at an angle of 45^0 is caught by an outfielder at a height of 3 feet.
a. How far was the ball hit from home plate?
b. What is the maximum height of the ball?
c. How fast was the ball traveling when it impacts the outfielder's glove?

Solution: Part a.) For the vector function describing projectile motion, we have

$$\mathbf{r}(t) = (v_0 \cos\alpha)t\ \mathbf{i} + [h + (v_0 \sin\alpha)t - \frac{1}{2}gt^2]\,\mathbf{j}$$

For this problem, the initial speed is $v_0 = 98$, the angle of elevation is $\alpha = 45^0$, and the initial height of the baseball is $h = 4$. Thus, since $\cos\alpha = \cos 45^0 = \frac{\sqrt{2}}{2}$, $\sin\alpha = \sin 45^0 = \frac{\sqrt{2}}{2}$, and $g = 32$, then the vector function becomes

$$\mathbf{r}(t) = (98 \cdot \frac{\sqrt{2}}{2})t\ \mathbf{i} + [4 + (98 \cdot \frac{\sqrt{2}}{2})t - \frac{1}{2}(32)t^2]\,\mathbf{j}.$$

Simplifying, this becomes

$$\mathbf{r}(t) = 49\sqrt{2}\,t\ \mathbf{i} + [4 + 49\sqrt{2}\,t - 16t^2]\,\mathbf{j}.$$

Hence, the parametric equation for the horizontal distance the ball travels is $x = 49\sqrt{2}\,t$ and the parametric equation for the height is $y = 4 + 49\sqrt{2}\,t - 16t^2$. To find how far the ball has traveled when it is caught, we must first find find the time *t* when the ball is caught. We know the ball is caught at a height of 3 feet. If we set $y = 3$, we obtain the equation

$$3 = 4 + 49\sqrt{2}\,t - 16t^2$$

This is a quadratic equation. To solve, we first get all terms on the left hand side of the equation and obtain

$$16t^2 - 49\sqrt{2}\,t - 1 = 0$$

To find *t*, we use the quadratic formula (we want to solve the equation of the form $at^2 + bt + c = 0$) given by

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

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Setting $a = 16$, $b = -49\sqrt{2}$, and $c = -1$, we obtain

$$t = \frac{-(-49\sqrt{2}) \pm \sqrt{(-49\sqrt{2})^2 - 4(16)(-1)}}{2(16)}$$

Noting that $(-49\sqrt{2})^2 = 2401 \cdot 2 = 4802$, we simplify this equation and get

$$t = \frac{49\sqrt{2} \pm \sqrt{4802 + 64}}{32}$$

or

$$t = \frac{49\sqrt{2} \pm \sqrt{4866}}{32}.$$

Converting to decimal, we see that

$$t \approx \frac{69.3 \pm 69.8}{32}$$

Thus, $t \approx \frac{69.3 - 69.8}{32} \approx -0.015$ and $t \approx \frac{69.3 + 69.8}{32} \approx 4.3$. Since the only practical solution is the positive result, we see the ball is caught after approximately $t = 4.3$ seconds. Since the parametric equation $x = 49\sqrt{2}\, t$ measures how far the ball has traveled horizontally after leaving the bat, we see the outfielder catches the ball after

$$\boxed{x = 49\sqrt{2}\, t = 49\sqrt{2}\,(4.3) \approx 298 \text{ feet}}$$

Part b.) To find the maximum height, we want to find the time *t* where the height parametric equation $y = 4 + 49\sqrt{2}\, t - 16t^2$ is maximized. We do this using a basic concept involving finding the critical number of this equation, that is, by finding the value of *t* where $y' = 49\sqrt{2} - 32t = 0$. Solving this equation for *t* gives $t = 49\sqrt{2}/32 \approx 2.2 \sec$. Hence, the maximum height is

$$\boxed{\textit{Maximum Height} = y(2.2) = 4 + 49\sqrt{2}\,(2.2) - 16(2.2)^2 \approx 79 \textit{ feet}}$$

Part c.) To find the speed when the ball is caught, we must first find an equation for the speed of the ball after time *t*. To do this, we first find the velocity. Taking the position function

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$$\mathbf{r}(t) = 49\sqrt{2}\,t\ \ \mathbf{i} + [4 + 49\sqrt{2}\,t - 16t^2]\,\mathbf{j}$$

we see that the velocity is

$$\mathbf{v}(t) = \mathbf{r}'(t) = 49\sqrt{2}\ \ \mathbf{i} + [49\sqrt{2} - 32t]\,\mathbf{j}\,.$$

Then, the equation for the speed is

$$\textit{Speed} = |\,\mathbf{v}(t)\,| = \sqrt{(49\sqrt{2})^2 + [49\sqrt{2} - 32t]^2} = \sqrt{4802 + [49\sqrt{2} - 32\,t]^2}$$

From part a, we saw that the ball is caught after $t = 4.3$ seconds. Thus, the speed when the ball is caught at this time is

$$\boxed{\underset{(t\,=\,4.3)}{\textit{Speed when ball is caught}} = |\,\mathbf{v}(4.3)\,| = \sqrt{4802 + [49\sqrt{2} - 32(4.3)]^2} \approx \sqrt{9647.4} \approx 97.3\ \ ft/\text{sec}}$$

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