Section 11.2: Limits and Continuity

Practice HW from Stewart Textbook (not to hand in)
p. 755 # 5-15 odd, 25-31 odd

Limits of Functions of Two Variables

A limit of a function of two variables $z = f(x, y)$ as (x, y) approaches a specific ordered pair.

Notation: We write

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

For a limit to exist, the function of 2 variables $z = f(x, y)$ must approach the same z value as (x, y) approaches (a, b) along all paths on the x-y coordinate plane. If a function of two variables is defined at a point, we can immediately substitute to find the limit.

Example 1: Evaluate $\lim_{(x,y)\to(1,0)} 5x + 3xy + y^2 + 1$, if it exists, or show that the limit does not exist.

Solution:

Notes:

1. To show that a limit does note exist we must produce two paths on the x-y coordinate plane that does not give the same limit value.
2. To show a limit does exist, sometimes the Squeeze Theorem. For functions of one variable, this says that if $f(x) \le g(x) \le h(x)$ when x approaches a (not necessarily when x equals a and
$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$$
then $\lim_{x\to a} g(x) = L$. This idea can be applied to functions of two variables.

Example 2: Evaluate $\lim_{(x,y)\to(0,0)} \dfrac{y^2}{x^2+y^2}$, if it exists, or show that the limit does not exist.

Solution:

$\blacksquare$

Example 3: Evaluate $\lim_{(x,y)\to(0,0)} \frac{6x^2 y}{2x^4 + y^4}$, if it exists, or show that the limit does not exist.

Solution:

Example 4: Evaluate $\lim_{(x,y)\to(0,0)} \frac{x^2 \sin^2 y}{x^2 + 2y^2}$, if it exists, or show that the limit does not exist.

Solution:

$\blacksquare$

Continuity

A function of two variables $f(x, y)$ is continuous at (a, b) if

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$

Note: To find where a function of 2 variables is continuous, often it suffices to find where the function is defined.

Example 5: Determine the set of points where the function $f(x,y) = \dfrac{x}{\sqrt{x+y}}$ is continuous.

Solution:

x

∎

Example 6: Determine the set of points where the function

$$f(x,y) = \begin{cases} \dfrac{x^2 \sin^2 y}{x^2 + 2y^2}, & if \quad (x,y) \neq (0,0) \\ \\ 2 & if \quad (x,y) = (0,0) \end{cases}$$

is continuous.

Solution: On this problem, the only point in question where the function may not be continuous at is the point (0, 0). We see first that f is defined at this point since $f(0,0) = 2$. From example 4, we see that $\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \dfrac{x^2 \sin^2 y}{x^2 + 2y^2} = 0$.

However, since

$$0 = \lim_{(x,y)\to(0,0)} f(x,y) \neq f(0,0) = 2$$

by the definition, the function is not continuous at the point (0, 0). This, the function is continuous for the following set:

$$\boxed{f \text{ is continuous for the set } \{ (x,y) \mid (x,y) \neq (0,0) \}}$$

$\blacksquare$