Section 11.3: Partial Derivatives

Practice HW from Stewart Textbook (not to hand in)
p. 767 # 5, 9, 13-37 odd, 47-52 odd

Partial Derivatives

Given a function of two variables $z = f(x, y)$. Then

$$\text{Partial Derivative with respect to } x = f_x(x, y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

$$\text{Partial Derivative with respect to } y = f_y(x, y) = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}$$

Notations For Partial Derivatives

Given $z = f(x, y)$.

$$\frac{\partial}{\partial x} f(x, y) = f_x(x, y) = \frac{\partial f}{\partial x} = \frac{\partial z}{\partial x} = \text{partial derivative with respect to } x.$$

$$\frac{\partial}{\partial y} f(x, y) = f_y(x, y) = \frac{\partial f}{\partial y} = \frac{\partial z}{\partial y} = \text{partial derivative with respect to } y.$$

$$\left.\begin{array}{l} \left.\dfrac{\partial z}{dx}\right|_{(a,b)} = f_x(a,b) \\[2ex] \left.\dfrac{\partial z}{dy}\right|_{(a,b)} = f_y(a,b) \end{array}\right\} \text{partial derivatives evaluated at the point } (a,b)$$

***Note:** In partial differentiation, we treat every variable as a constant except for the one we are differentiating with respect to.

Example 1: Find the first partial derivatives of the function $f(x, y) = 1 - x^2 - y^2$.

Solution:

∎

Example 2: Find the first partial derivatives of the function $f(x, y) = 4x^3y^2 + 5x^3 + x^2e^y$.

Solution:

∎

Example 3: Find the first partial derivatives of the function $f(x, y) = \ln(x^2 + y^2) + e^{x^2+y^2}$.

Solution:

$\blacksquare$

Example 4: Find $f_x(1,1)$ and $f_y(1,1)$ for $f(x, y) = \dfrac{xy}{x^2 + y^2}$

Solution:

Note: We can also differentiate functions of more than 2 variables.

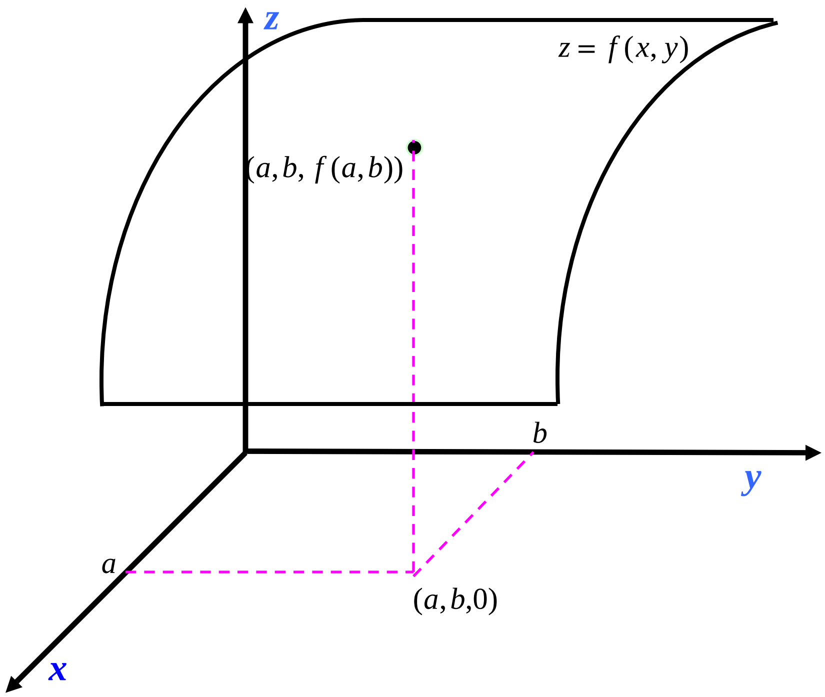
Example 5: Find the first partial derivatives of the function

$$f(x, y, z) = \sqrt{x^2 + y^2 + z^2} + \sin(x^2 + y^2) \, .$$

Solution:

Geometric Interpretation of the Partial Derivative

Given a surface $z = f(x, y)$. We want to consider the point ($a, b, f(a, b)$)



$$f_x(a,b) = \left.\frac{\partial z}{\partial x}\right|_{(a,b)} = \text{Slope of the tangent line to the surface at the point } (a,b,f(a,b)) \text{ in the } x \text{ direction}$$

$$f_y(a,b) = \left.\frac{\partial z}{\partial y}\right|_{(a,b)} = \text{Slope of the tangent line to the surface at the point } (a,b,f(a,b)) \text{ in the } y \text{ direction}$$

Example 6: Find the slope of the surface $f(x, y) = x^2 + y^2$ at the point (-2, 1, 5).

Solution:

∎

Second Order Partial Derivatives

Just as we can find second order derivatives for functions of one variable, we can do the same for functions of two variables.

Notation for Second Order Derivatives

$f_{xx}(x,y) = \frac{\partial}{dx}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial z^2}$: Take partial with respect to *x*, then with respect to *x* again

$f_{yx}(x,y) = \frac{\partial}{dy}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 z}{\partial y \partial x}$: Take partial with respect to *x*, then with respect to *y*.

$f_{xy}(x,y) = \frac{\partial}{dx}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 z}{\partial x \partial y}$: Take partial with respect to *y*, then with respect to *x*.

$f_{yy}(x,y) = \frac{\partial}{dy}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial y^2}$: Take partial with respect to *y*, then with respect to *y* again

Note: In general,

$$f_{xy}(x,y) = f_{yx}(x,y)$$

Example 7: Find all of the second derivatives for $f(x, y) = 3xy^2 + 2xy + x^2$.

Solution:

∎