# Section 11.6: Directional Derivatives and the Gradient Vector

Practice HW from Stewart Textbook (not to hand in)
p. 778 # 1-4
p. 799 # 4-15, 17, 19, 21, 29, 35, 37 odd

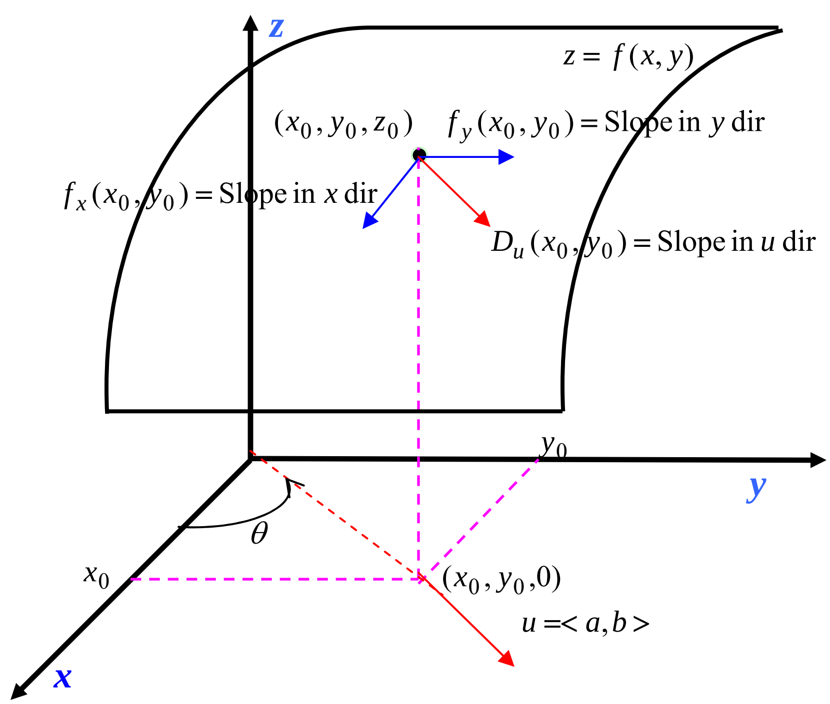
## The Directional Derivative

Recall that

$$f_x(a,b) = \left.\frac{\partial z}{\partial x}\right|_{(a,b)} = \begin{array}{c}\text{Slope of the tangent line to the surface at the} \\ \text{point } (a, b, f(a,b)) \text{ in the } x \text{ direction}\end{array}$$

$$f_y(a,b) = \left.\frac{\partial z}{\partial y}\right|_{(a,b)} = \begin{array}{c}\text{Slope of the tangent line to the surface at the} \\ \text{point } (a, b, f(a,b)) \text{ in the } y \text{ direction}\end{array}$$

Instead of restricting ourselves to the *x* and *y* axis, suppose we want to find a method for finding the slope of the surface in any desired direction.



Let $\boldsymbol{u}$ = < *a, b* > be the unit vector (a vector of length one) on the *x*-*y* plane which indicates the direction we are moving. Then we define the following:

**Definition of the Directional Derivative**

The directional derivative of a function $z = f(x, y)$ in the direction of the unit vector $\boldsymbol{u} = \langle a, b \rangle$, denoted by $D_{\mathbf{u}} f(x,y)$, is defined the be the following:

$$D_{\mathbf{u}} f(x,y) = f_x(x,y)a + f_y(x,y)b$$

**Notes**

1. Geometrically, the directional derivative is used to calculate the slope of the surface $z = f(x, y)$. That is, to calculate the slope of the surface at the point $(x_0, y_0, z_0)$, where $z_0 = f(x_0, y_0)$, we compute the following:

$$\begin{matrix} \textit{Slope of Surface at point } (x_0, y_0, z_0) \\ \textit{in direction of unit vector } \mathbf{u} = \langle a.b \rangle \end{matrix} = D_{\mathbf{u}} f(x_0, y_0) = f_x(x_0, y_0)a + f_y(x_0, y_0)b$$

2. The vector $\boldsymbol{u} = \langle a, b \rangle$ must be a unit vector. If we want to compute the directional derivative of a function in the direction of the vector **v** and **v** is not a unit vector, we compute

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{1}{|\mathbf{v}|}\mathbf{v}.$$

3. The direction of the unit vector **u** can be expressed in terms of the angle $\theta$ between the vector **u** and the *x*-axis. In this case, $\mathbf{u} = \langle \cos\theta, \sin\theta \rangle$ (note, **u** is a unit vector since $|\mathbf{u}| = \sqrt{\cos^2\theta + \sin^2\theta} = \sqrt{1} = 1$) and the directional derivative can be expressed as

$$D_{\mathbf{u}} f(x,y) = f_x(x,y)\cos\theta + f_y(x,y)\sin\theta.$$

4. Computationally, the directional derivative represents the rate of change of the function $f$ in the direction of the unit vector **u**.

**Example 1:** Find the directional derivative of the function $f(x, y) = 3y - 4xy + 6x$ at the point (1, 2) in the direction of the unit vector that makes an angle of $\theta = \dfrac{\pi}{3}$ radians with the $x$-axis.

**Solution:**

$\blacksquare$

**Example 2:** Find the directional derivative of the function $f(x, y) = 3y - 4xy + 6x$ at the point (-3, -4) in the direction of the vector $\mathbf{v} = -2\mathbf{i} + 3\mathbf{j}$.

**Solution:**

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## Gradient of a Function

Given a function of two variables $z = f(x, y)$, the gradient vector, denoted by $\nabla f(x, y)$, is a vector in the *x*-*y* plane denoted by

$$\nabla f(x,y) = f_x(x,y)\,\mathbf{i} + f_y(x,y)\,\mathbf{j}$$

**Facts about Gradients**

1. The directional derivative of the function $z = f(x, y)$ in the direction of the unit vector $\boldsymbol{u} = < a, b >$ can be expressed in terms of gradient using the dot product. That is,

$$\begin{aligned} D_{\mathbf{u}} f(x,y) &= \nabla f(x,y) \cdot \mathbf{u} \\ &= < f_x(x,y), f_y(x,y) > \cdot < a,b > \\ &= f_x(x,y)a + f_y(x,y)b \end{aligned}$$

2. The gradient vector $\nabla f(x,y)$ gives the direction of maximum increase of the surface $z = f(x, y)$. The length of the gradient vector is the maximum value of the directional derivative (the maximum rate of change of *f*). That is,

$$\textit{Maximum Value of the Directional Derivative } D_{\mathbf{u}} f(x,y) = |\nabla f(x,y)|$$

3. The negation of the gradient vector $-\nabla f(x,y)$ gives the direction of maximum decrease. of the surface $z = f(x, y)$. The negation of the length of the gradient vector is the minimum value of the directional derivative. That is,

$$\textit{Minimum Value of the Directional Derivative } D_{\mathbf{u}} f(x,y) = -|\nabla f(x,y)|$$

**Example 3:** Given the function $f(x, y) = y\cos(x - y)$.

a. Find the gradient of $f$

b. Evaluate the gradient at the point $P(\frac{\pi}{3}, 0)$.

c. Use the gradient to find a formula for the directional derivative of $f$ in the direction of the vector $\mathbf{u} = < -\frac{3}{5}, \frac{4}{5} >$. Use the result to result to find the rate of change of $f$ at P in the direction of the vector **u**.

**Solution:**

**Directional Derivative and Gradient for Functions of 3 variables**

The directional derivative of a function $f(x, y, z)$ of 3 variables in the direction of the unit vector $\boldsymbol{u} = < a, b, c >$, denoted by $D_{\mathbf{u}} f(x, y, z)$, is defined to be the following:

$$D_{\mathbf{u}} f(x,y,z) = f_x(x,y,z)a + f_y(x,y,z)b + f_z(x,y,z)c$$

The gradient vector, denoted by $\nabla f(x,y,z)$, is a vector denoted by

$$\nabla f(x,y,z) = f_x(x,y,z)\,\mathbf{i} + f_y(x,y,z)\,\mathbf{j} + f_z(x,y,z)\,\mathbf{k}$$

**Example 4:** Find the gradient and directional derivative of $f(x,y,z) = 5x^2 - 3xy + xyz$ at P(1, 2, 4) in the direction of the point Q(-3, 1, 2).

**Solution:** We first compute the first order partial derivatives with respect to *x*, *y*, and *z*. They are as follows.

$$f_x(x,y,z) = 10x - 3y(1) + yz(1) = 10x - 3y + yz$$
$$f_y(x,y,z) = 0 - 3x(1) + xz(1) = -3x + xz$$
$$f_z(x,y,z) = 0 - 0 + xy(1) = xy\,.$$

Then the formula for the gradient is computed as follows:

$$\nabla f(x,y,z) = f_x(x,y,z)\,\mathbf{i} + f_y(x,y,z)\,\mathbf{j} + f_z(x,y,z)\,\mathbf{k} = (10x - 3y + yz)\,\mathbf{i} + (-3x + xz)\,\mathbf{j} + xy\,\mathbf{k}$$

Hence, at the point P(1, 2, 4), the gradient is

$$\boxed{\nabla f(1,2,4) = (10(1) - 3(2) + (2)(4))\,\mathbf{i} + (-3(1) + (1)(4))\,\mathbf{j} + (1)(2)\,\mathbf{k} = 12\,\mathbf{i} + \mathbf{j} + 2\,\mathbf{k} = <12,1,2>}$$

To find the directional derivative, we must first find the unit vector **u** specifying the direction at the point P(1, 2, 4) in the direction of the point Q(-3, 1, 2). To do this, we find the vector $\mathbf{v} = \overrightarrow{\mathbf{PQ}}$. This is found to be $\mathbf{v} = \overrightarrow{\mathbf{PQ}} = < -3 - 1, 1 - 2, 2 - 4 > = < -4, -1, -2 >$. This must be a unit vector, so we compute the following:

$$\mathbf{u} = \frac{\mathbf{1}}{|\,\mathbf{v}\,|}\mathbf{v} = \frac{1}{\sqrt{(-4)^2 + (-1)^2 + (-2)^2}} < -4,-1,-2 > = \frac{1}{\sqrt{21}} < -4,-1,-2 > = < -\frac{4}{\sqrt{21}}, -\frac{1}{\sqrt{21}} - \frac{2}{\sqrt{21}} >$$

Then, using the dot product formula involving the gradient for the directional derivative and the results for the gradient at the point P(1,2,4) and **u** given above, we obtain

$$\begin{aligned}
\begin{matrix} \textit{The directional derivative} \\ \textit{at the point } P(1,24) \end{matrix} &= D_{\mathbf{u}} f(1,2,4) = \nabla f(1,2,4) \cdot \mathbf{u} \\
&= < 12,1,2 > \cdot < -\frac{4}{\sqrt{21}}, -\frac{1}{\sqrt{21}}, -\frac{2}{\sqrt{21}} > \\
&= (12)(-\frac{4}{\sqrt{21}}) + (1)(-\frac{1}{\sqrt{21}}) + 2(-\frac{2}{\sqrt{21}}) \\
&= -\frac{48}{\sqrt{21}} - \frac{1}{\sqrt{21}} - \frac{4}{\sqrt{21}} \\
&= \boxed{-\frac{53}{\sqrt{21}} \approx -11.6}
\end{aligned}$$

**Example 5:** Find the maximum rate of change of $f(x, y, z) = 5x^2 - 3xy + xyz$ at the point (1, 2, 4) and the direction in which it occurs.

**Solution:**

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## Normal Lines to Surfaces

Recall that $z = f(x, y)$ gives a 3D surface in space. We want to form the following functions of 3 variables

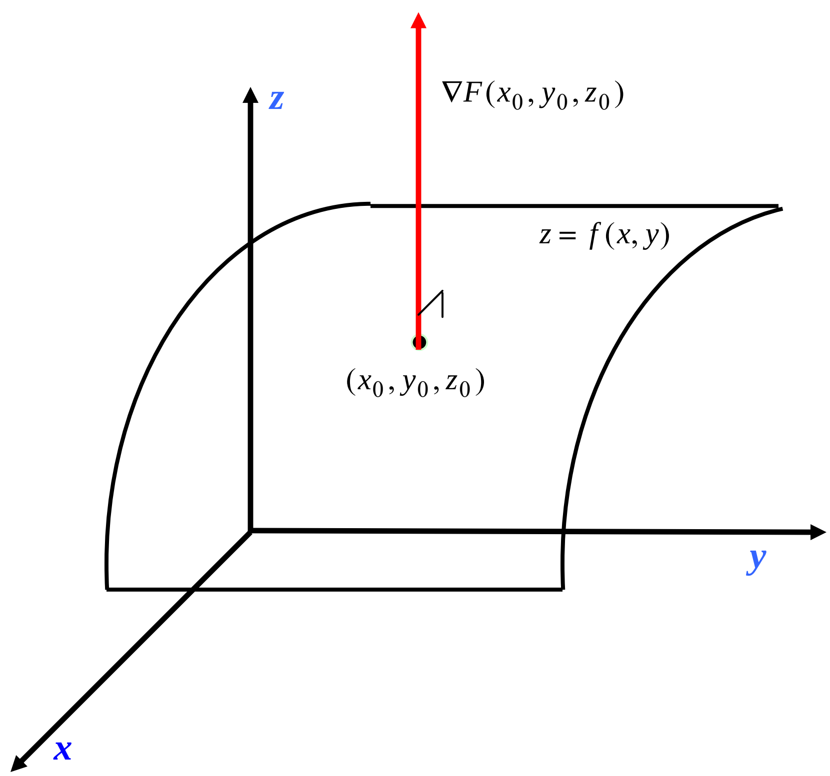
$$F(x, y, z) = f(x, y) - z$$

Note that the function $F$ is obtained by moving all terms to one side of an equation and setting them equal to zero. We use the following basic fact.

**Fact**: Given a point $(x_0, y_0, z_0)$ on a surface, the gradient of $F$ at this point

$$\nabla F(x_0, y_0, z_0) = F_x(x_0, y_0, z_0)\,\mathbf{i} + F_y(x_0, y_0, z_0)\,\mathbf{j} + F_z(x_0, y_0, z_0)\,\mathbf{k}$$

is a vector orthogonal (normal) to the surface $z = f(x, y)$.



**Example 6:** Find a unit normal vector to the surface $x^2 + y^2 + z^2 = 9$ at the point (2, 1, 2)

**Solution:**

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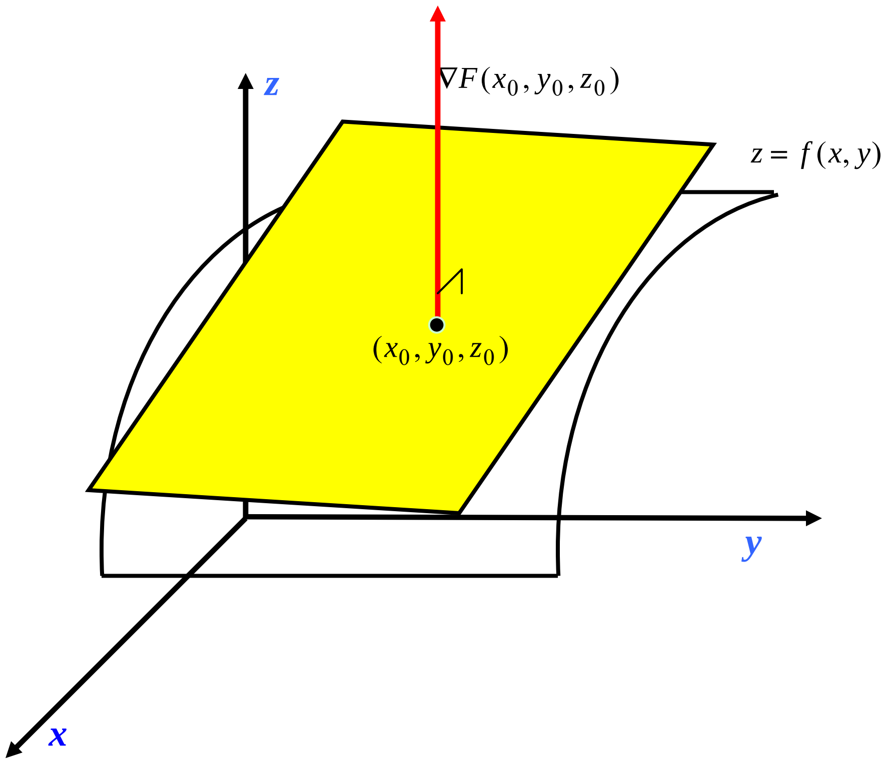
## Tangent Planes

Using the gradient, we can find a equation of a plane tangent to a surface and a line normal to a surface. Consider the following:




Recall that to write equation of a plane, we need a point on the plane and a normal vector. Since $\nabla F(x_0, y_0, z_0) = < F_x(x_0, y_0, z_0), F_y(x_0, y_0, z_0), F_z(x_0, y_0, z_0) >$ represents a normal vector to the surface (and the tangent plane), its components can be used to write the equation of the tangent plane at the point $(x_0, y_0, z_0)$. The equation of the tangent plane is given as follows:

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0 .$$

Recall, to write the equation of a line in 3D space, we need a point and a parallel vector. Since $\nabla F(x_0, y_0, z_0) = < F_x(x_0, y_0, z_0), F_y(x_0, y_0, z_0), F_z(x_0, y_0, z_0) >$ is a vector normal to the surface, it would be parallel to any line normal to the surface at $(x_0, y_0, z_0)$. Thus, the parametric equations of the normal line are:

$$x = x_0 + F_x(x_0, y_0, z_0)t , \ y = y_0 + F_y(x_0, y_0, z_0)t , \ z = z_0 + F_z(x_0, y_0, z_0)t$$

We summarize these results as follows.

**Tangent Plane and Normal Line Equations to a Surface**

Given a surface $z = f(x, y)$ in 3D, form the function $F(x, y, z) = f(x, y) - z$ of three variables. Then the equation of the tangent plane to the surface $z = f(x, y)$ at the point $(x_0, y_0, z_0)$ is given by

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0 .$$

The parametric equations of the normal line through the point $(x_0, y_0, z_0)$ are given by

$$x = x_0 + F_x(x_0, y_0, z_0)t ,\ y = y_0 + F_y(x_0, y_0, z_0)t ,\ z = z_0 + F_z(x_0, y_0, z_0)t$$

**Note:** Recall that to find the symmetric equations of a line, take the parametric equations, solve for *t*, and set the results equal.

**Example 7:** Find the equation of the tangent plane and the parametric and symmetric equations for the normal line to the surface $x^2 + y^2 + z^2 = 9$ at the point (2, 1, 2).

**Solution:**

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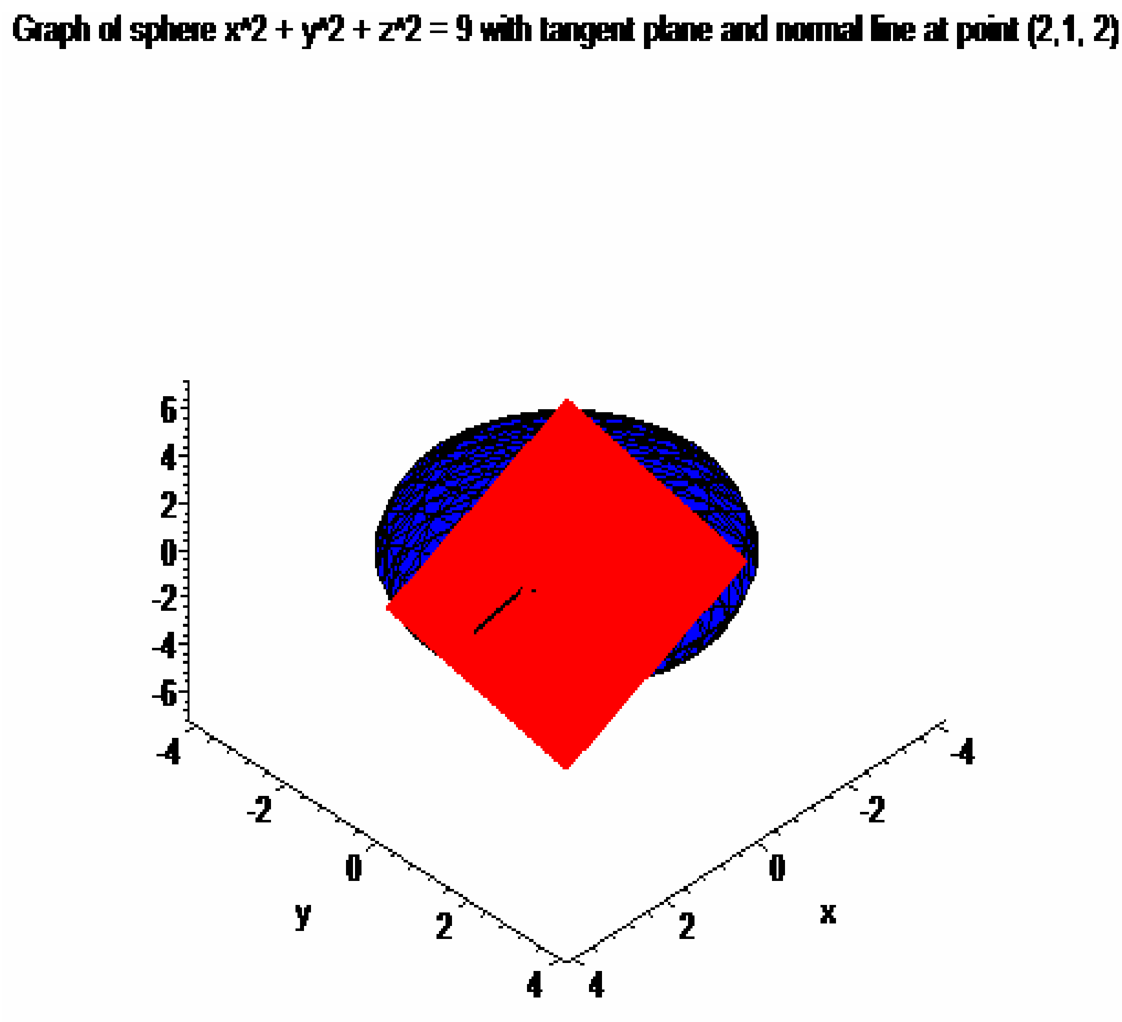
**Note:** The following graph using Maple shows the graph of the sphere $x^2 + y^2 + z^2 = 9$ with the tangent plane and normal line at the point (2, 1, 2).



**Example 8:** Find the equation of the tangent plane and the parametric and symmetric equations for the normal line to the surface $f(x,y) = e^{-x^2-y^2}$ at the point $(0,1,\frac{1}{e})$.

**Solution:** We start by setting $z = e^{-x^2-y^2}$ and computing the function of 3 variables

$$F(x,y,z) = f(x,y) - z = e^{-x^2-y^2} - z$$

Recall that to get an equation of any plane, including a tangent plane, we need a point and a normal vector. We are given the point $(0,1,\frac{1}{e})$. The normal vector comes from computing the gradient vector of $F$ at this point. Recall that for a given point $(x_0, y_0, z_0)$, the gradient vector at this point is given by the formula

$$\nabla F(x_0,y_0,z_0) = F_x(x_0,y_0,z_0)\,\mathbf{i} + F_y(x_0,y_0,z_0)\,\mathbf{j} + F_z(x_0,y_0,z_0)\,\mathbf{k}$$

Computing the necessary partial derivatives, we obtain

$$F_x(x,y,z) = -2xe^{-x^2-y^2} - 0 = -2xe^{-x^2-y^2}$$
$$F_y(x,y,z) = -2ye^{-x^2-y^2} - 0 = -2ye^{-x^2-y^2}$$
$$F_z(x,y,z) = 0 - 1 = -1$$

The given point is $(x_0, y_0, z_0) = (0,1,\frac{1}{e})$. Thus, since

$$F_x(x_0,y_0,z_0) = F_x(0,1,\frac{1}{e}) = -2(0)e^{-(0)^2-(1)^2} = 0\,,$$
$$F_y(x_0,y_0,z_0) = F_y(0,1,\frac{1}{e}) = -2(1)e^{-(0)^2-(1)^2} = -2e^{-1} = -\frac{2}{e}, \text{ and}$$
$$F_z(x_0,y_0,z_0) = F(0,1,\frac{1}{e}) = -1\,,$$

the gradient vector of $F$ at the point $(0,1,\frac{1}{e})$ is

$$\nabla F(0,1,\frac{1}{e}) = 0\,\mathbf{i} + (1)\,\mathbf{j} + \frac{1}{e}\,\mathbf{k} = \mathbf{j} + \frac{1}{e}\,\mathbf{k}$$

We use the components of the gradient vector to write the equation of the tangent plane using the formula

**(continued on next page)**

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$$

At the point $(x_0, y_0, z_0) = (0, 1, \frac{1}{e})$, this formula becomes

$$F_x(0, 1, \frac{1}{e})(x - 0) + F_y(0, 1, \frac{1}{e})(y - 1) + F_z(0, 1, \frac{1}{e})(z - \frac{1}{e}) = 0$$

Using the calculations for the partial derivatives given on the previous page, this equation becomes

$$(0)(x - 0) + (-\frac{2}{e})(y - 1) + (-1)(z - \frac{1}{e}) = 0$$

or

$$-\frac{2}{e}(y - 1) - (z - \frac{1}{e}) = 0$$

We can expand this equation to get it in general form. Doing this gives

$$-\frac{2}{e}y + \frac{2}{e} - z + \frac{1}{e} = 0$$

and when combining like terms, we have the equation of the tangent plane

$$\boxed{-\frac{2}{e}y - z + \frac{3}{e} = 0.}$$

The parametric equations of the normal line through the point $(x_0, y_0, z_0)$ are given by

$$x = x_0 + F_x(x_0, y_0, z_0)t, \ y = y_0 + F_y(x_0, y_0, z_0)t, \ z = z_0 + F_z(x_0, y_0, z_0)t$$

Using the calculations we computed above where that $(x_0, y_0, z_0) = (0, 1, \frac{1}{e})$,

$F_x(x_0, y_0, z_0) = F_x(0, 1, \frac{1}{e}) = 0$, $F_y(x_0, y_0, z_0) = F_y(0, 1, \frac{1}{e}) = -\frac{2}{e}$, and

$F_z(x_0, y_0, z_0) = F(0, 1, \frac{1}{e}) = -1$, we obtain

$$x = 0 + (0)t, \ y = 1 + (-\frac{2}{e})t, \ z = \frac{1}{e} + (-1)t$$

which, when simplified, gives **(continued on next page)**

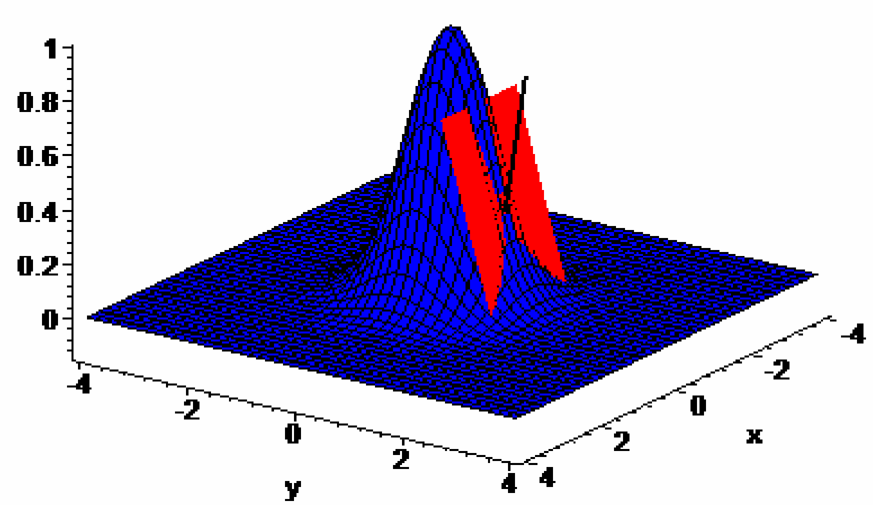
$$x = 0\,,\ y = 1 - \frac{2}{e}t\,,\ z = \frac{1}{e} - t$$

If we want to convert this these equations to symmetric form, we can take the last two equations of the previous result and solve for *t*. This gives $t = \dfrac{y-1}{-2/e}$ and $t = \dfrac{y-1/e}{-1}$. Equation gives the symmetric equations of the normal line.

$$\frac{y-1}{-2/e} = \frac{y-1/e}{-1}$$

The following displays the graph of the function $f(x,y) = e^{-x^2-y^2}$, the tangent plane, and the normal line at the point $(0,1,\frac{1}{e})$.

Graph of surface z = e^(-x^2-y^2) with tangent plane and normal line at point (0, 1, 1/e)

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