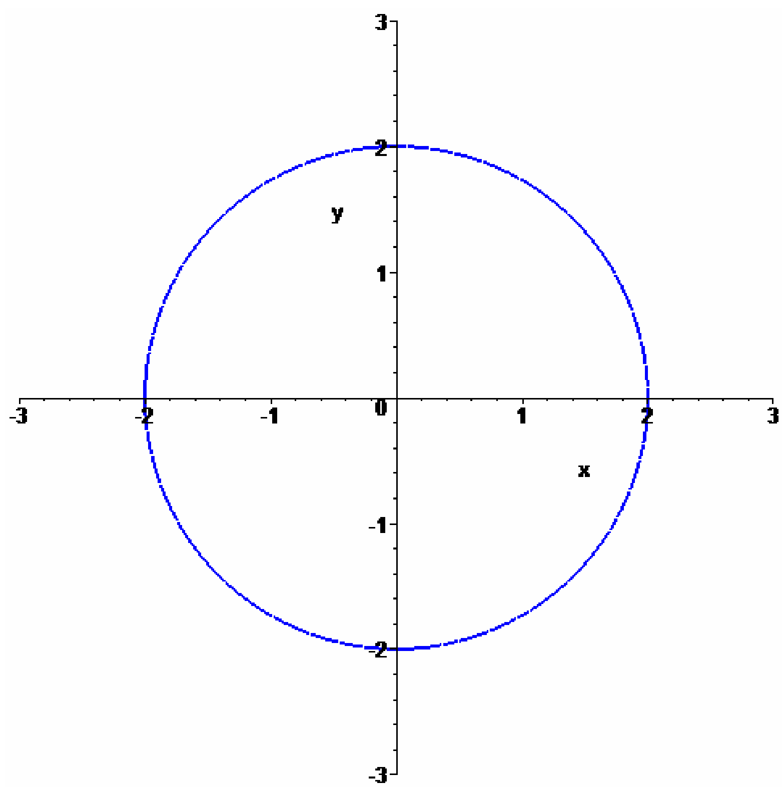
Section 12.4: Double Integrals in Polar Coordinates

Practice HW from Stewart Textbook (not to hand in)
p. A66 Appendix H: # 1-6
p. 856 Section 12.4: # 1-21 odd, 25, 27 odd

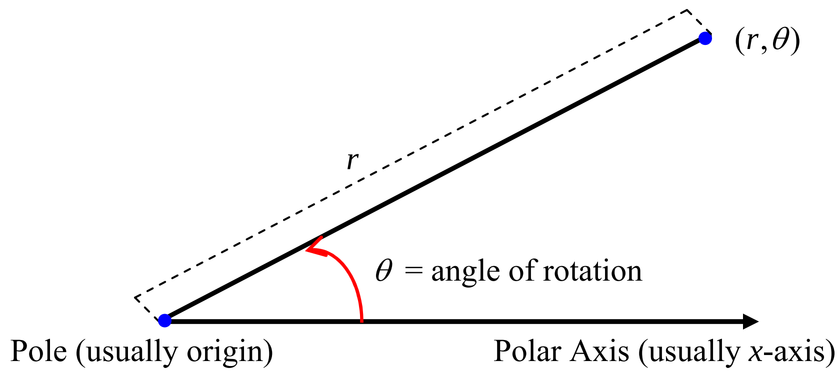
Polar Coordinates

Up to now , we have represented graphs as a collection of points (x, y) in the rectangular coordinate. For example, the following represents the graph of the circle $x^2 + y^2 = 4$ in rectangular coordinates.

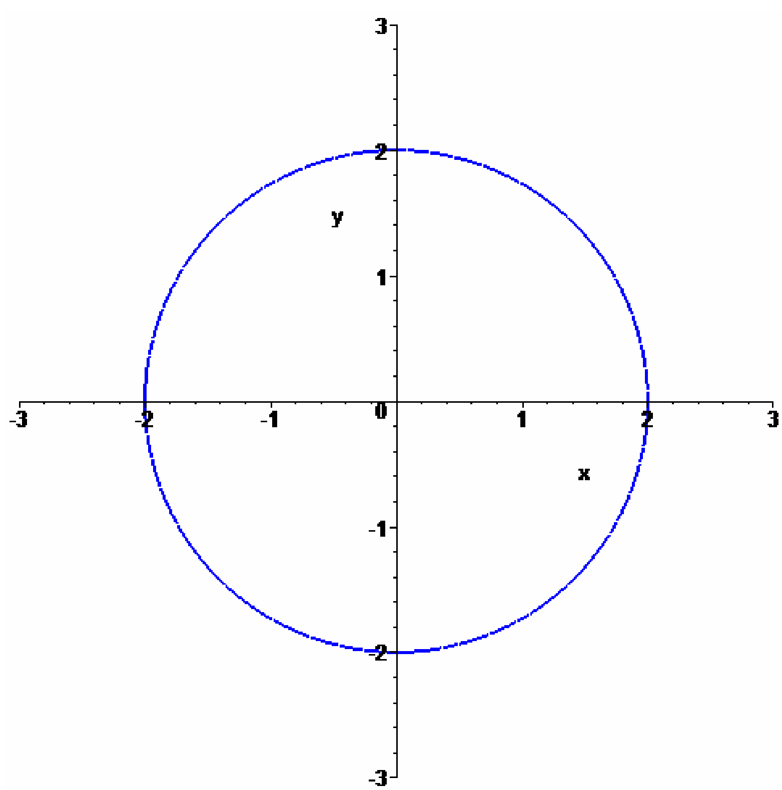


Equations like this can be expressed in *polar coordinates*.

In polar coordinates, each coordinate is of the form (r, θ)



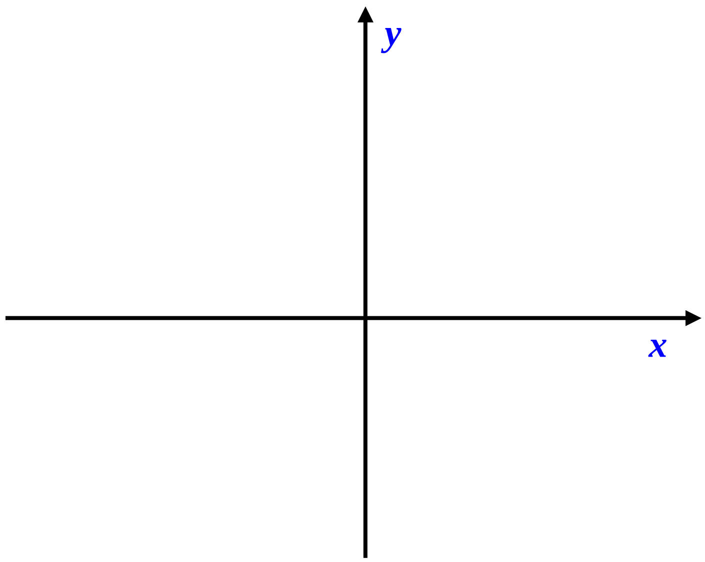
In polar coordinates, for the circle $x^2 + y^2 = 4$, the points on the circle have a different representation.



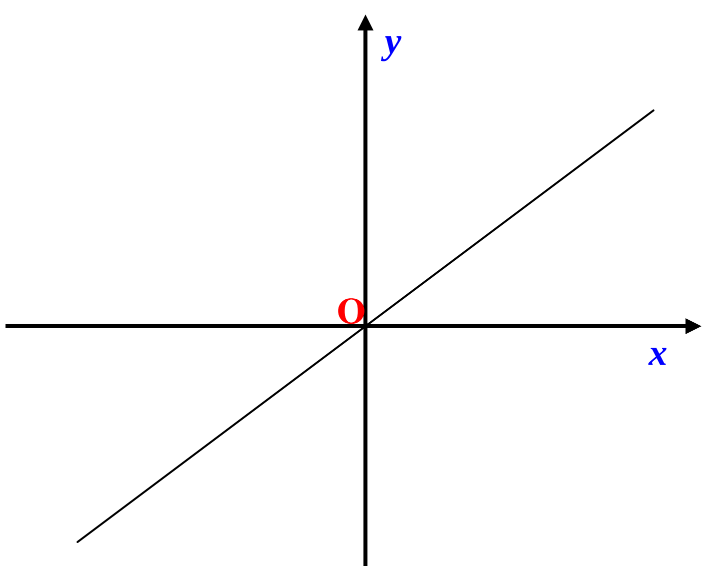
Note: Polar Coordinates are not unique – there may be more than one way to represent the same point.

In general, (r, θ) and $(r, \theta + 2n\pi)$, where n is an integer, give the same point.

For example, $(2,\frac{\pi}{2})$ and $(2,\frac{\pi}{2}+2\pi)=(2,\frac{5\pi}{2})$ represent the same point. Also, $(2,\pi)$ and $(2,3\pi)$ represent the same point.



Note: r can also be negative. The points (r,θ) and $(-r,\theta)$ lie on the line same line through the pole O and the same distance $|r|$ from O, but on opposites sides of O. The points $(r,\theta+\pi)$ and $(-r,\theta)$ represent the same point.



Example 1: Plot the points with polar coordinates $(2, \frac{\pi}{3})$, $(-2, \frac{\pi}{3})$, $(1, \frac{5\pi}{4})$, and $(-3, \frac{11\pi}{6})$.

Solution:

∎

Example 2: Plot the point with polar coordinates $(4, \pi)$. Then find two other pairs of polar coordinates of this point, one with $r > 0$ and the other $r < 0$.

Solution:

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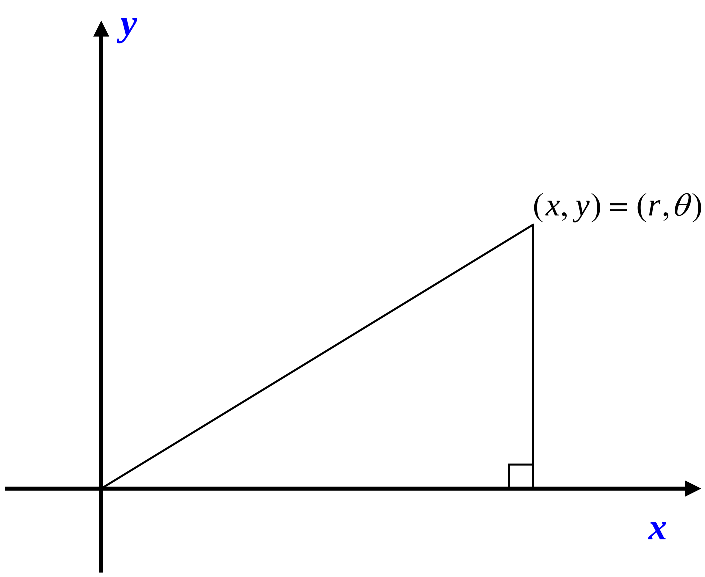
Conversion of Rectangular and Polar Coordinates

Consider the following diagram:



We say $\cos\theta = \dfrac{\text{adjacent}}{\text{hypotenuse}} = \dfrac{x}{r}$, $\sin\theta = \dfrac{\text{opposite}}{\text{hypotenuse}} = \dfrac{y}{r}$, and $\sin\theta = \dfrac{\text{opposite}}{\text{adjacent}} = \dfrac{y}{x}$.

Using these equations and the Pythagorean Theorem, we have the following conversion equations.

Conversion Formulas

To convert from polar form (r, θ) to rectangular form (x, y) and vise versa, we use the following conversion equations.

From polar to rectangular form: $x = r\cos\theta$ and $y = r\sin\theta$.

From rectangular to polar form: $r^2 = x^2 + y^2$ and $\tan\theta = \dfrac{y}{x}$.

Example 3: Find the corresponding rectangular coordinates for the point $(1, \frac{5\pi}{4})$.

Solution:

∎

Example 4: Find the polar coordinates for the point (0, -5).

Solution:

∎

Converting Equations

Example 5: Convert the equation $x = 10$ to polar form.

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Example 5: Convert the equation $x^2 + y^2 - 2x = 0$ to polar form.

Graphing Polar Equations

One way to graph polar equations is to convert it to rectangular form and sketch the rectangular equation.

Example 6: Convert *r* = 3 to rectangular form and sketch the graph.

Solution:

Example 7: Convert $r = 2\sec\theta$ to rectangular form and sketch the graph.

Solution:

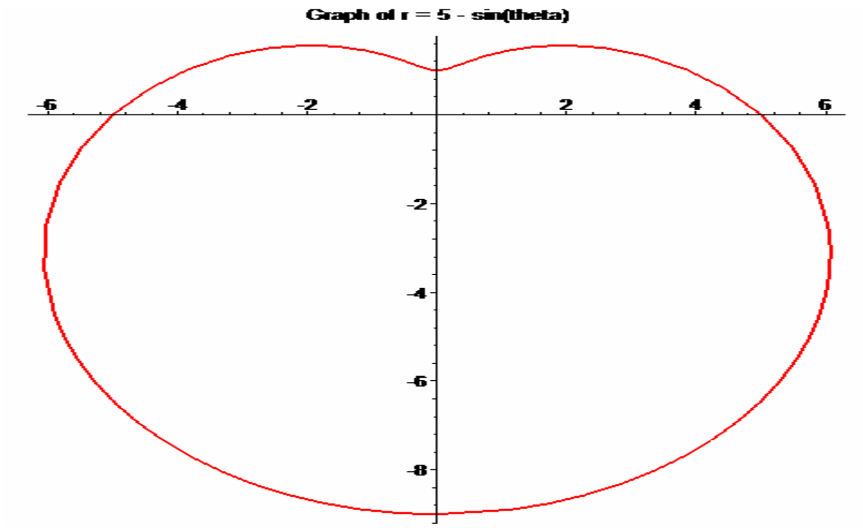
$\blacksquare$

Note: In general, sketching graphs is polar form is not an easy task. Maple can be a useful tool in graphing. The following shows the Maple commands necessary to graph the polar graphs $r = 5 - 4\sin\theta$ and $r = 2\cos(3\theta)$ (next page)

```
> with(plots):
> r := 5 - 4*sin(theta);
```

$$r := 5 - 4\,\sin(\theta)$$

```
> polarplot(r, theta = 0..2*Pi, thickness = 2, title =
"Graph of r = 5 - sin(theta)");
```




```
> r := 2*cos(3*theta);
```

$$r := 2\,\cos(3\,\theta)$$

```
> polarplot(r, theta = 0..2*Pi, thickness = 2, title =
"Graph of r = 2cos(3*theta)");
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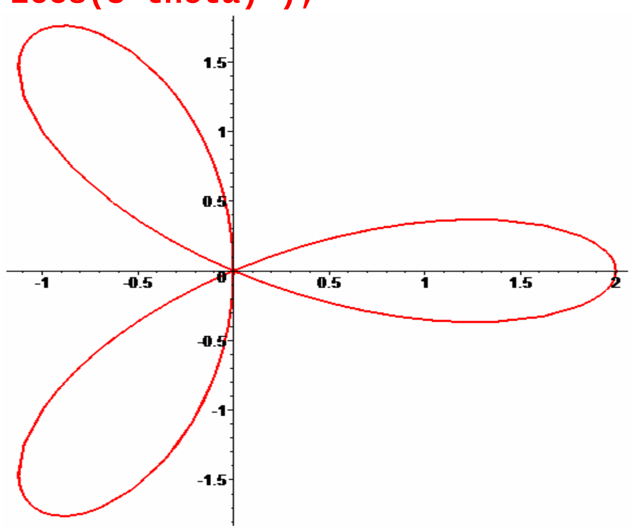
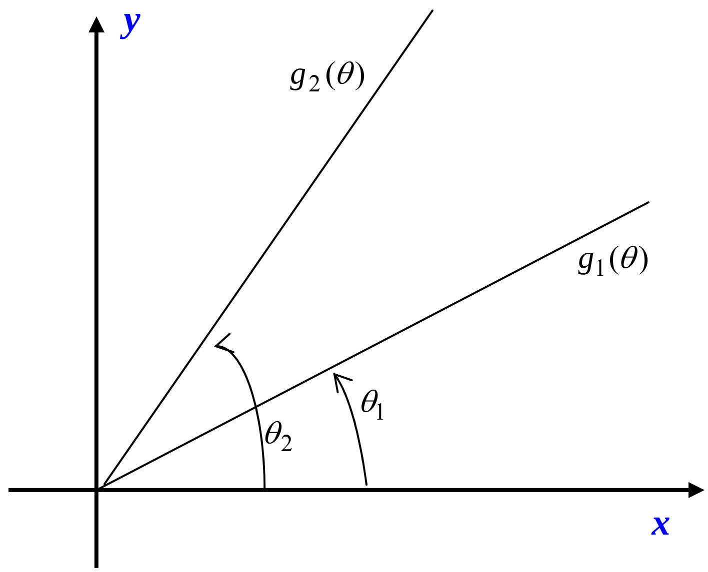


Evaluating Double Integrals Using Polar Coordinates

Changing a double integral from rectangular to polar coordinates can sometimes result in an integral that is easier to evaluate.

Suppose we have a region *R* on the *x*-*y* plane satisfying the polar conditions

$$0 \le g_1(\theta) \le r \le g_2(\theta) \text{ and } \theta_1 \le \theta \le \theta_2 .$$



Then if the function of two variables $z = f(x, y)$ is defined over *R*, we say that

$$\begin{matrix}\text{Volume under} \\ z = f(x,y) \ge 0 \text{ over R}\end{matrix} = \iint_R f(x,y)\, dA = \int_{\theta=\theta_1}^{\theta=\theta_2} \int_{r=g_1(\theta)}^{r=g_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

Example 8: Use polar coordinates to evaluate $\iint_R (x+y)\, dA$ where R is the region that lies I in the first quadrant between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

Solution:

Example 9: Find the volume under the surface $z = e^{-x^2-y^2}$ and above the disk $x^2 + y^2 \leq 4$.

Solution:

$\blacksquare$

Example 10: Evaluate the iterated integral $\int_{0}^{2}\int_{y}^{\sqrt{8-y^2}} \sqrt{x^2 + y^2}\, dx\, dy$

Solution:

■