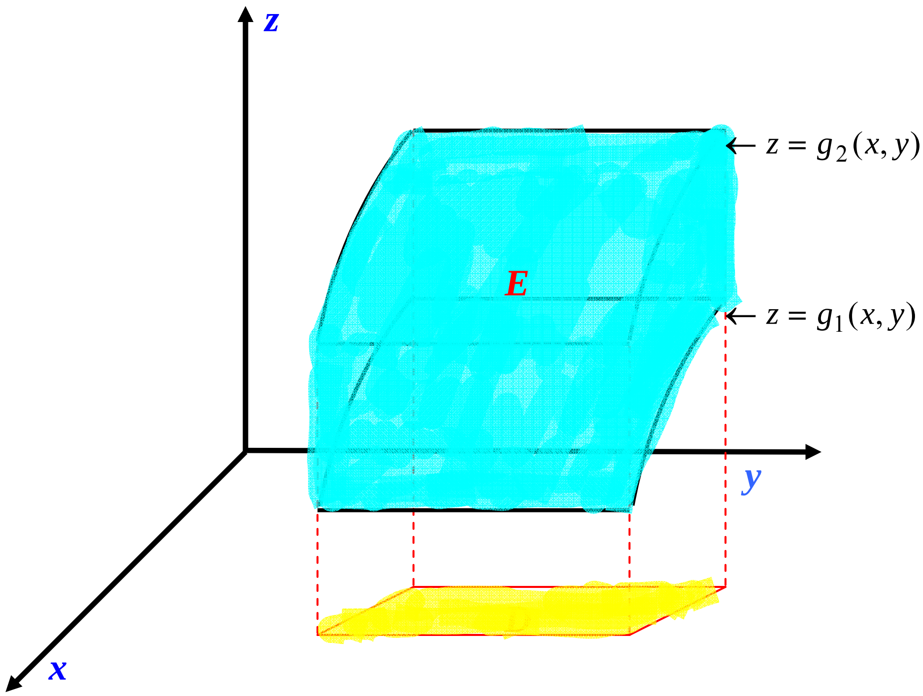
Section 12.7: Triple Integrals

Practice HW from Stewart Textbook (not to hand in)
p. 879 # 1-19 odd

Consider a continuous function of 3 variables $f(x, y, z)$ on the solid bounded region E on the 3D plane.



Then

$$\begin{array}{c}\text{The Triple Integral} \\ \text{of } f\end{array} = \iiint_E f(x,y,z)\,dV$$

$$\begin{array}{c}\text{Volume of the} \\ \text{Region Q}\end{array} = \iiint_E dV$$

We evaluate a triple integral by writing it as an integrated integral.

$$\iiint_E f(x,y,z)\,dV = \int_{x=a}^{x=b} \int_{y=h_1(x)}^{y=h_2(x)} \int_{z=g_1(x,y)}^{z=g_2(x,y)} f(x,y,z)\,dz\,dy\,dx$$

where the limits of integration are defined on the boundaries of E.

Example 1: Evaluate the iterated integral $\int_{1}^{4}\int_{1}^{e^2}\int_{0}^{\frac{1}{xz}} \ln z \; dy\, dz\, dx$

Solution:

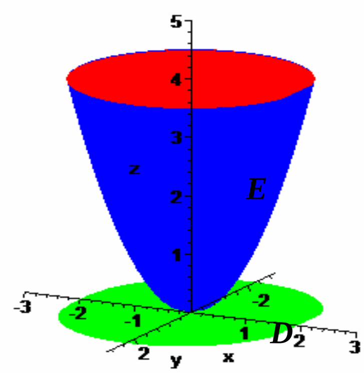
Example 2: Evaluate $\iiint_E 2xy$ where E lies under the plane $x + y + z = 10$ and above the region in the *x*-*y* plane bounded by the curves in the first octant $y = 0$ and $y = \sqrt{16 - x^2}$.

Solution:

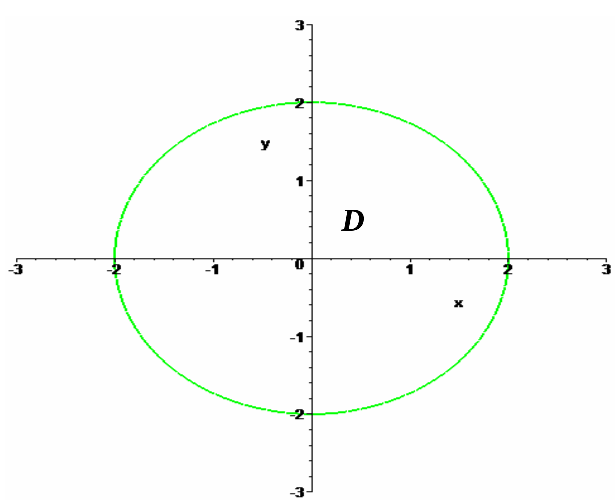
[redacted]

Example 3: Use a triple integral to find the volume of the solid bounded by the graphs of $z = x^2 + y^2$ and the plane $z = 4$.

Solution: The following graph shows a plot of the paraboloid $z = x^2 + y^2$ (in blue), the plane $z = 4$ (in red), and its projection onto the *x*-*y* plane (in green).



The triple integral $\iiint_E dV$ will evaluate the volume of this surface. In the *z* direction, the surface *E* is bounded between the graphs of the paraboloid $z = x^2 + y^2$ and the plane $z = 4$. This will make up the limits of integration in terms of *z*. The limits for *y* and *x* are determined by looking at the projection *D* given on the *x*-*y* plane, which is the graph of the circle $x^2 + y^2 = 4$ given as follows:



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Taking the equation $x^2 + y^2 = 4$ and solving for *y* gives $y = \pm\sqrt{4 - x^2}$. Thus the limits of integration of *y* will range from $y = -\sqrt{4 - x^2}$ to $y = \sqrt{4 - x^2}$. The integration limits in terms of *x* hence range from *x* = -2 to *x* = 2. Thus the volume of the region *E* can be found by evaluating the following triple integral:

Volume of $E = \iiint_E dV = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{x^2+y^2}^{4} dz\, dy\, dx$. If we evaluate the intermost integral we get the following:

$$\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{x^2+y^2}^{4} dz\, dy\, dx = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} [z]\Big|_{z=x^2+y^2}^{z=4}\ dy\, dx$$

$$= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left[4 - (x^2 + y^2)\right] dy\, dx$$

Since the limits involving *y* involve two radicals, integrating the rest of this result in rectangular coordinates is a tedious task. However, since the region *D* on the *x*-*y* plane given by $x^2 + y^2 = 4$ is circular, it is natural to represent this region in polar coordinates. Using the fact that the radius *r* ranges from $r = 0$ to $r = 2$ and that θ ranges from $\theta = 0$ to $\theta = 2\pi$ and also that in polar coordinates, the conversion equation is $r^2 = x^2 + y^2$, the iterated integral becomes

$$\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left[4 - (x^2 + y^2)\right] dy\, dx = \int_{0}^{2\pi} \int_{0}^{2} (4 - r^2) r\, dr\, d\theta$$

Evaluating this integral in polar coordinates, we obtain

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$$\int_0^{2\pi}\int_0^2 (4-r^2)r\,dr\,d\theta = \int_0^{2\pi}\int_0^2 (4r-r^3)\,dr\,d\theta \qquad \text{(Distribute } r\text{)}$$

$$= \int_{\theta=0}^{\theta=2\pi} (2r^2 - \frac{1}{4}r^4)\Big|_{r=0}^{r=2} d\theta \qquad \text{(Integrate)}$$

$$= \int_{\theta=0}^{\theta=2\pi} [(2(2)^2 - \frac{1}{4}(2)^4) - 0]\,d\theta \qquad \text{(Sub in limits of integration)}$$

$$= \int_{\theta=0}^{\theta=2\pi} 4\,d\theta \qquad \text{(Simplify)}$$

$$= 4\theta\Big|_{\theta=0}^{\theta=2\pi} \qquad \text{(Integrate)}$$

$$= 4(2\pi) - 4(0) \qquad \text{(Sub in limits of integration)}$$

$$= 8\pi$$

Thus, the volume of E is 8π .

$\blacksquare$