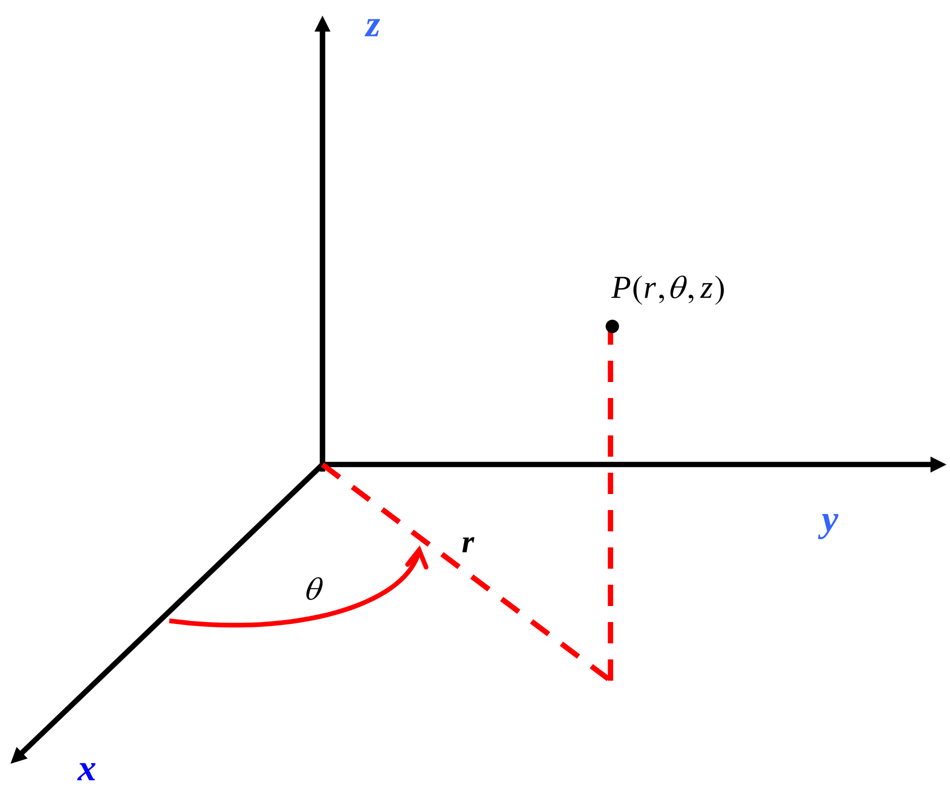
Section 9.7/12.8: Triple Integrals in Cylindrical and Spherical Coordinates

Practice HW from Stewart Textbook (not to hand in)
Section 9.7: p. 689 # 3-23 odd
Section 12.8: p. 887 # 1-11 odd, 13a, 17-21 odd, 23a, 31, 33

Cylindrical Coordinates

Cylindrical coordinates extend polar coordinates to 3D space. In the cylindrical coordinate system, a point *P* in 3D space is represented by the ordered triple (r, θ, z). Here, *r* represents the distance from the origin to the projection of the point *P* onto the *x*-*y* plane, θ is the angle in radians from the *x* axis to the projection of the point on the *x*-*y* plane, and *z* is the distance from the *x*-*y* plane to the point *P*.



As a review, the next page gives a review of the sine, cosine, and tangent functions at basic angle values and the sign of each in their respective quadrants.

Sine and Cosine of Basic Angle Values

θ **Degrees**	θ **Radians**	$\cos\theta$	$\sin\theta$	$\tan\theta = \frac{\sin\theta}{\cos\theta}$
0	0	$\cos 0 = 1$	$\sin 0 = 0$	0
30	$\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{3}$
45	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
60	$\frac{\pi}{3}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\sqrt{3}$
90	$\frac{\pi}{2}$	0	1	undefined
180	π	-1	0	0
270	$\frac{3\pi}{2}$	0	-1	undefined
360	2π	1	0	0

Signs of Basic Trig Functions in Respective Quadrants

Quadrant	$\cos\theta$	$\sin\theta$	$\tan\theta = \frac{\sin\theta}{\cos\theta}$
I	+	+	+
II	-	+	-
III	-	-	+
IV	+	-	-

The following represent the conversion equations from cylindrical to rectangular coordinates and vice versa.

Conversion Formulas

To convert from cylindrical coordinates (r, θ, z) to rectangular form (x, y, z) and vise versa, we use the following conversion equations.

From polar to rectangular form: $x = r\cos\theta$, $y = r\sin\theta$, $z = z$.

From rectangular to polar form: $r^2 = x^2 + y^2$, $\tan\theta = \frac{y}{x}$, and $z = z$

Example 1: Convert the points $(\sqrt{2}, \sqrt{2}, 3)$ and $(-3, \sqrt{3}, -1)$ from rectangular to cylindrical coordinates.

Solution:

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Example 2: Convert the point $(3, -\frac{\pi}{4}, 1)$ from cylindrical to rectangular coordinates.

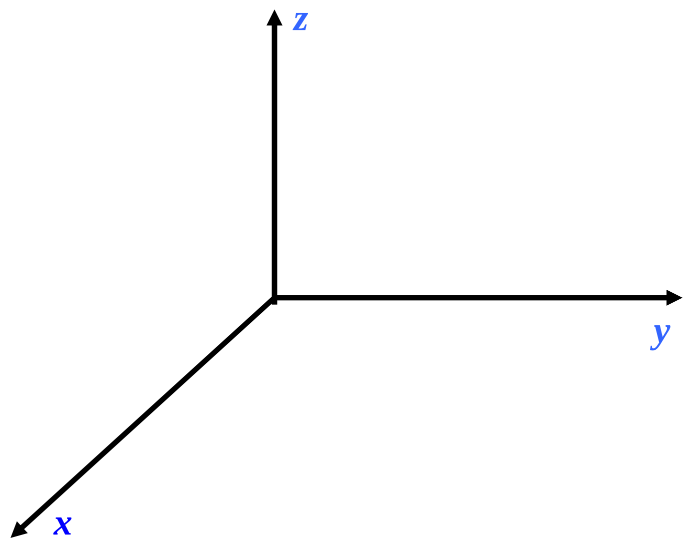
Solution:

Graphing in Cylindrical Coordinates

Cylindrical coordinates are good for graphing surfaces of revolution where the *z* axis is the axis of symmetry. One method for graphing a cylindrical equation is to convert the equation and graph the resulting 3D surface.

Example 3: Identify and make a rough sketch of the equation $z = r^2$.

Solution:



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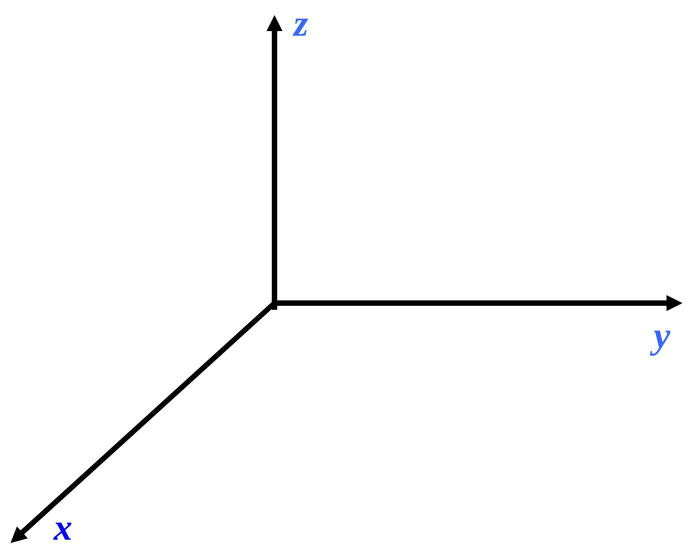
Example 4: Identify and make a rough sketch of the equation $\theta = \frac{\pi}{4}$.

Solution:



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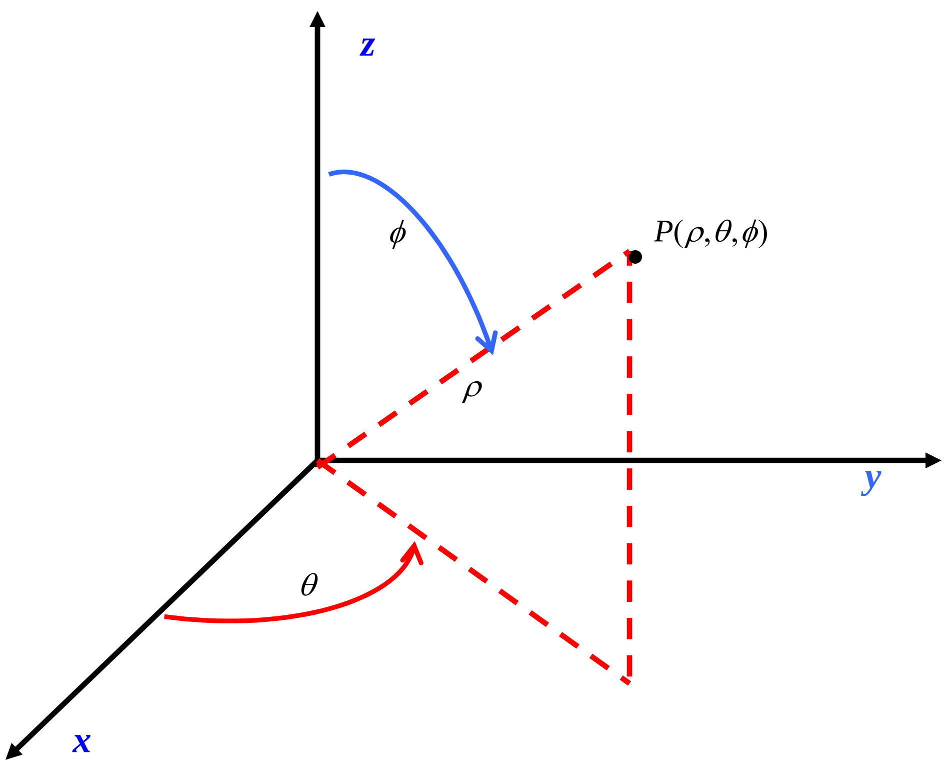
Spherical Coordinates

Spherical coordinates represents points from a spherical "global" perspective. They are good for graphing surfaces in space that have a point or center of symmetry.

Points in spherical coordinates are represented by the ordered triple

$$(\rho, \theta, \phi)$$

where ρ is the distance from the point to the origin O, θ, where is the angle in radians from the *x* axis to the projection of the point on the *x*-*y* plane (same as cylindrical coordinates), and ϕ is the angle between the positive *z* axis and the line segment $\overrightarrow{OP}$ joining the origin and the point $\mathrm{P}(\rho, \theta, \phi)$. Note $0 \le \phi \le \pi$.



Conversion Formulas

To convert from cylindrical coordinates (ρ, θ, ϕ) to rectangular form (*x, y, z*) and vise versa, we use the following conversion equations.

From to rectangular form: $x = \rho \sin\phi \ \cos\theta$, $y = \rho \sin\phi \ \sin\theta$, $z = \rho \cos\phi$

From rectangular to polar form: $\rho^2 = x^2 + y^2 + z^2$, $\tan\theta = \dfrac{y}{x}$, and

$$\phi = \arccos\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right) = \arccos\left(\frac{z}{\rho}\right)$$

Example 5: Convert the points (1, 1, 1) and $(-3, -\sqrt{3}, 2\sqrt{2})$ from rectangular to spherical coordinates.

Solution:

Example 6: Convert the point $(9, \frac{\pi}{4}, \pi)$ from rectangular to spherical coordinates.

Solution:

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Example 7: Convert the equation $\rho = 2\sec\phi$ to rectangular coordinates.

Solution:

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Example 8: Convert the equation $\phi = \frac{\pi}{3}$ to rectangular coordinates.

Solution: For this problem, we use the equation $\phi = \arccos(\frac{z}{\sqrt{x^2 + y^2 + z^2}})$. If we take the cosine of both sides of the this equation, this is equivalent to the equation

$$\cos\phi = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$$

Setting $\phi = \frac{\pi}{3}$ gives

$$\cos\frac{\pi}{3} = \frac{z}{\sqrt{x^2 + y^2 + z^2}}.$$

Since $\cos\frac{\pi}{3} = \frac{1}{2}$, this gives

$$\frac{1}{2} = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$$

or

$$\boxed{\sqrt{x^2 + y^2 + z^2} = 2z}$$

Hence, $\sqrt{x^2 + y^2 + z^2} = 2z$ is the equation in rectangular coordinates. Doing some algebra will help us see what type of graph this gives.

Squaring both sides gives

$$x^2 + y^2 + z^2 = (2z)^2$$
$$x^2 + y^2 + z^2 = 4z^2$$
$$x^2 + y^2 - 3z^2 = 0$$

The graph of $x^2 + y^2 - 3z^2 = 0$ is a cone shape half whose two parts be found by graphing the two equations $\pm\sqrt{x^2 + y^2 + z^2} = 2z$. The graph of the top part, $\sqrt{x^2 + y^2 + z^2} = 2z$, is displayed as follows on the next page.

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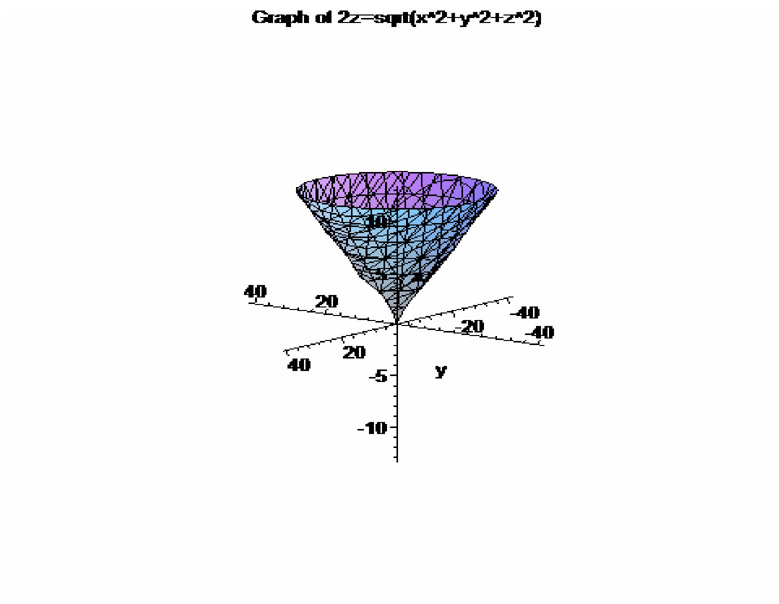

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Example 9: Convert the equation $x^2 + y^2 = z$ to cylindrical coordinates and spherical coordinates.

Solution: For cylindrical coordinates, we know that $r^2 = x^2 + y^2$. Hence, we have $r^2 = z$ or

$$\boxed{r = \pm\sqrt{z}}$$

For spherical coordinates, we let $x = \rho\sin\phi\ \cos\theta$, $y = \rho\sin\phi\ \sin\theta$, and $z = \rho\cos\phi$ to obtain

$$(\rho\sin\phi\ \cos\theta)^2 + (\rho\sin\phi\ \sin\theta)^2 = \rho\cos\phi$$

We solve for ρ using the following steps:

$$\rho^2\sin^2\phi\ \cos^2\theta + \rho^2\sin^2\phi\ \sin^2\theta = \rho\cos\phi \qquad \text{(Square terms)}$$

$$\rho^2\sin^2\phi(\cos^2\theta + \sin^2\theta) = \rho\cos\phi \qquad \text{(Factor } \rho^2\sin^2\phi\text{)}$$

$$\rho^2\sin^2\phi\ (1) - \rho\cos\phi = 0 \qquad \text{(Use identity } \cos^2\theta + \sin^2\theta = 1\text{)}$$

$$\rho(\rho\sin^2\phi - \cos\phi) = 0 \qquad \text{(Factor } \rho\text{)}$$

$$\rho = 0, \rho\sin^2\phi - \cos\phi = 0 \qquad \text{(Set each factor equal to zero and solve)}$$

$$\boxed{\rho = 0, \rho = \frac{\cos\phi}{\sin^2\phi}}$$

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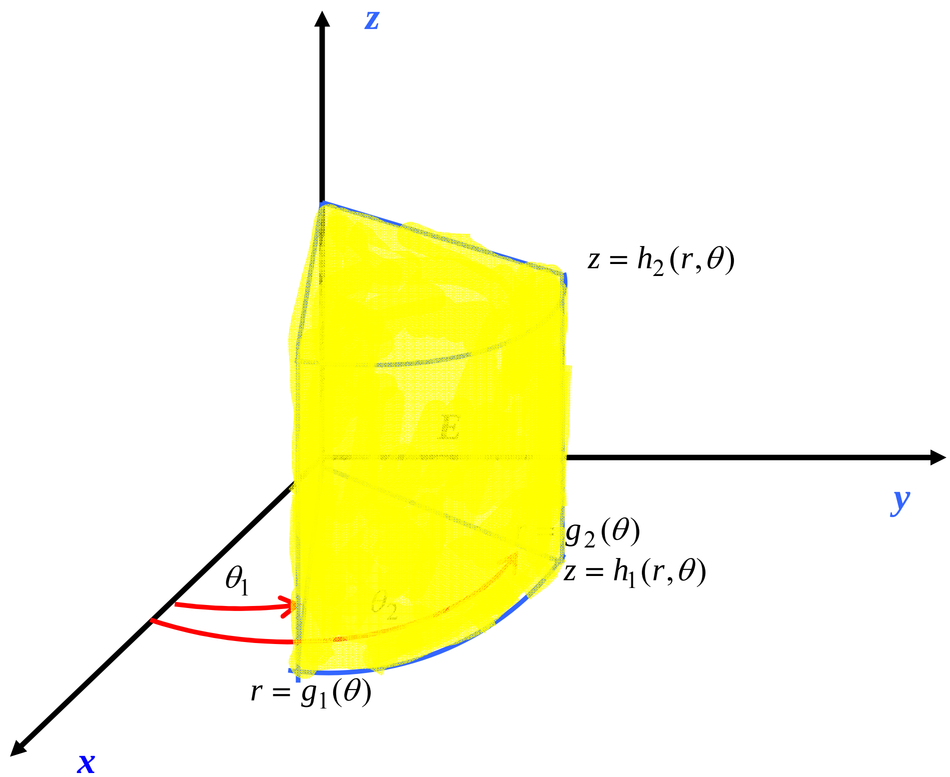
Triple Integrals in Cylindrical Coordinates

Suppose we are given a continuous function of three variables $f(r,\theta,z)$ expressed over a solid region E in 3D where we use the cylindrical coordinate system.



Then

$$\iiint_E f(r,\theta,z)\,dV = \int_{\theta=\theta_1}^{\theta=\theta_2} \int_{r=g_1(\theta)}^{r=g_2(\theta)} \int_{z=h_1(r,\theta)}^{z=h_2(r,\theta)} f(r,\theta,z)\ r\ dz\ dr\ d\theta$$

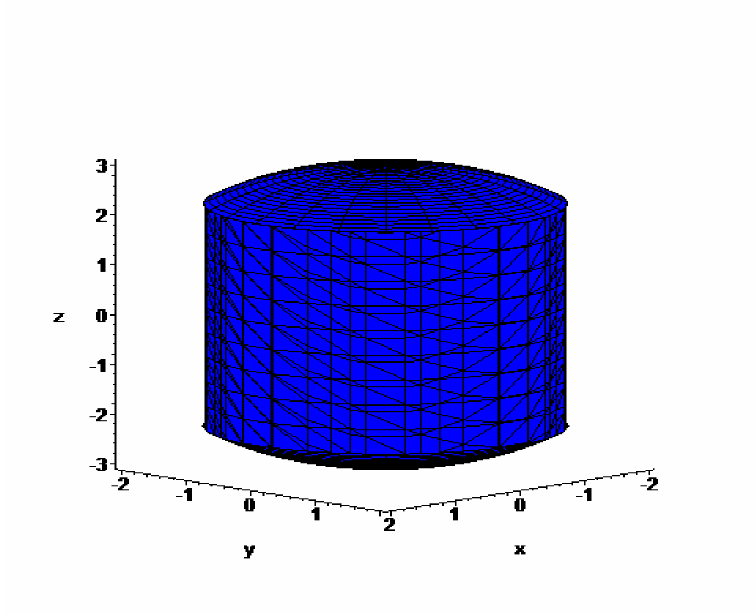
$$\text{Volume of E} = \iiint_E dV = \int_{\theta=\theta_1}^{\theta=\theta_2} \int_{r=g_1(\theta)}^{r=g_2(\theta)} \int_{z=h_1(r,\theta)}^{z=h_2(r,\theta)} r\ dz\ dr\ d\theta$$

Example 10: Use cylindrical coordinates to evaluate $\iiint\limits_E (x^3 + xy^2)\, dV$, where E is the solid in the first octant that lies beneath the paraboloid $z = 1 - x^2 - y^2$.

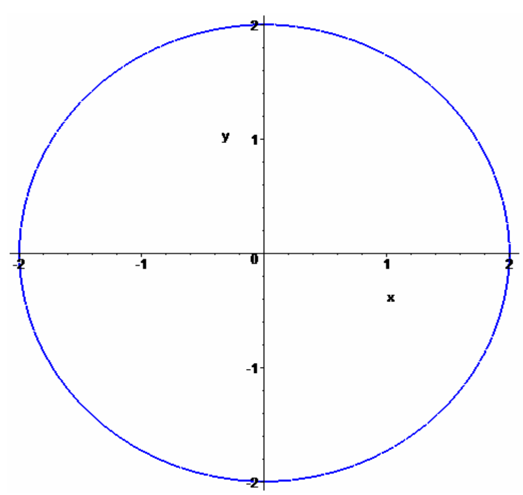
Solution:

Example 11: Use cylindrical coordinates to find the volume of the solid that lies both within the cylinder $x^2 + y^2 = 4$ and the sphere $x^2 + y^2 + z^2 = 9$.

Solution: Using Maple, we can produce the following graph that represents this solid:



In this graph, the shaft of the solid is represented by the cylinder equation $x^2 + y^2 = 4$. It is capped on the top and bottom by the sphere $x^2 + y^2 + z^2 = 9$. Solving for *z*, the upper and bottom portions of the sphere can be represented by the equations $z = \pm\sqrt{9 - x^2 - y^2}$. Thus, *z* ranges from $z = -\sqrt{9 - x^2 - y^2}$ to $z = \sqrt{9 - x^2 - y^2}$. Since $x^2 + y^2 = r^2$ in cylindrical coordinates, these limits become $z = -\sqrt{9 - r^2}$ to $z = \sqrt{9 - r^2}$.When this surface is projected onto the *x*-*y* plane, it is represented by the circle $x^2 + y^2 = 4$. The graph is



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This is a circle of radius 2. Thus, in cylindrical coordinates, this circle can be represented from $r = 0$ to $r = 2$ and from $\theta = 0$ to $\theta = 2\pi$. Thus, the volume can be represented by the following integral:

$$Volume = \iiint_E dV = \int_{\theta=\theta_1}^{\theta=\theta_2} \int_{r=g_1(\theta)}^{r=g_2(\theta)} \int_{z=h_1(r,\theta)}^{z=h_2(r,\theta)} r\,dz\ dr\ d\theta = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=2} \int_{z=-\sqrt{9-r^2}}^{z=\sqrt{9-r^2}} r\ dz\ dr\ d\theta$$

We evaluate this integral as follows:

$$\int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=2} \int_{z=-\sqrt{9-r^2}}^{z=\sqrt{9-r^2}} r\ dz\ dr\ d\theta = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=2} rz\Big|_{z=-\sqrt{9-r^2}}^{z=\sqrt{9-r^2}}\ dr\ d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=2} r(\sqrt{9-r^2}) - r(-\sqrt{9-r^2})\, dr\ d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=2} 2r\sqrt{9-r^2}\ dr\ d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi} -\frac{2}{3}(9-r^2)^{\frac{3}{2}}\Big|_{r=0}^{r=2}\ d\theta \qquad \text{(Use u - du sub let } u = 9 - r^2\text{)}$$

$$= \int_{\theta=0}^{\theta=2\pi} [-\frac{2}{3}(9-2^2)^{\frac{3}{2}} - -\frac{2}{3}(9-0^2)^{\frac{3}{2}}]\, d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi} [-\frac{2}{3}(5)^{\frac{3}{2}} + \frac{2}{3}(9)^{\frac{3}{2}}]\, d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi} [18 - \frac{10}{3}\sqrt{5}]\, d\theta \qquad \text{(Note } (9)^{\frac{3}{2}} = 27 \text{ and } (5)^{\frac{3}{2}} = 5\sqrt{5}\text{)}$$

$$= [18 - \frac{10}{3}\sqrt{5}]\theta\Big|_{\theta=0}^{\theta=2\pi}$$

$$= (18 - \frac{10}{3}\sqrt{5})2\pi - 0$$

$$\boxed{= 36\pi - \frac{20\pi}{3}\sqrt{5}}$$

Thus, the volume is $36\pi - \frac{20\pi}{3}\sqrt{5}$.

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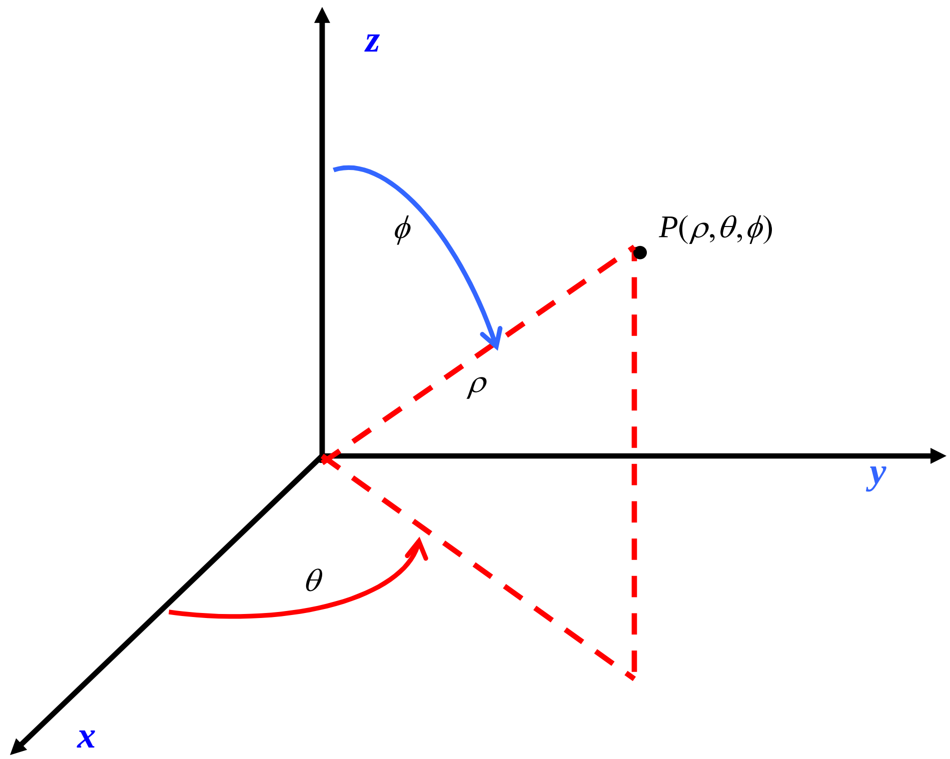
Triple Integrals in Spherical Coordinates

Suppose we have a continuous function $f(\rho,\phi,\theta)$ defined on a bounded solid region *E*.



Then

$$\iiint_E f(\rho,\phi,\theta)\,dV = \int_{\theta=\theta_1}^{\theta=\theta_2}\int_{\phi=\phi_1}^{\phi=\phi_2}\int_{\rho=h_1(\phi,\theta)}^{\rho=h_2(\phi,\theta)} f(\rho,\phi,\theta)\ \rho^2\sin\phi\ d\rho\ d\phi\ d\theta$$

$$\text{Volume of E} = \iiint_E dV = \int_{\theta=\theta_1}^{\theta=\theta_2}\int_{\phi=\phi_1}^{\phi=\phi_2}\int_{\rho=h_1(\phi,\theta)}^{\rho=h_2(\phi,\theta)} \rho^2\sin\phi\ d\rho\ d\phi\ d\theta$$

Example 12: Use spherical coordinates to evaluate $\iiint_E e^{\sqrt{x^2+y^2+z^2}}\, dV$, where E is enclosed by the sphere $x^2 + y^2 + z^2 = 9$ in the first octant.

Solution:

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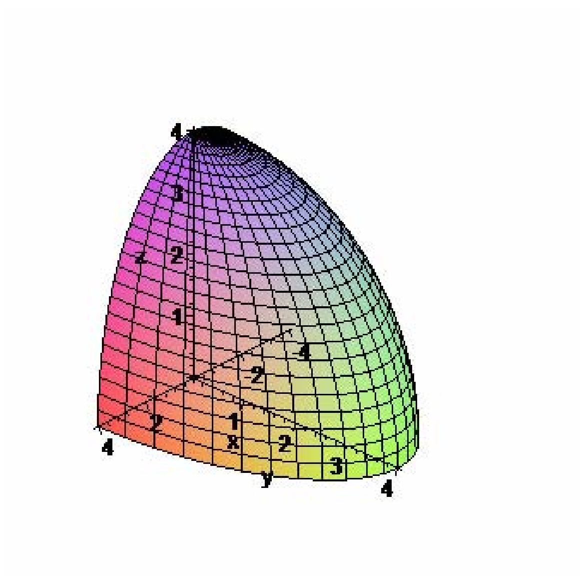
Example 13: Convert $\int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}}\int_{0}^{\sqrt{16-x^2-y^2}} \sqrt{x^2+y^2}\ dz\ dy\ d\theta$ from rectangular to spherical coordinates and evaluate.

Solution: Using the identities $x = \rho\sin\phi\cos\theta$ and $y = \rho\sin\phi\sin\theta$, the integrand becomes

$$\begin{aligned}\sqrt{x^2+y^2} &= \sqrt{\rho^2\sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta} \\ &= \sqrt{\rho^2\sin^2\phi(\cos^2\theta+\sin^2\theta)} \\ &= \sqrt{\rho^2\sin^2\phi(1)} = \rho\sin\phi\end{aligned}$$

The limits with respect to *z* range from $z = 0$ to $z = \sqrt{16-x^2-y^2}$. Note that $z = \sqrt{16-x^2-y^2}$ is a hemisphere and is the upper half of the sphere $x^2+y^2+z^2 = 16$. The limits with respect to *y* range from $y = 0$ to $y = \sqrt{4-x^2}$, which is the semicircle located on the positive part of the *y* axis on the *x*-*y* plane of the circle $x^2+y^2 = 4$ as *x* ranges from $x = -2$ to $x = 2$. Hence, the region described by these limits is given by the following graph



Thus, we can see that ρ ranges from $\rho = 0$ to $\rho = 4$, ϕ ranges from $\phi = 0$ to $\phi = \frac{\pi}{2}$ and θ ranges from $\theta = 0$ to $\theta = \pi$. Using these results, the integral can be evaluated in polar coordinates as follows:

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$$\int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}}\int_{0}^{\sqrt{16-x^2-y^2}} \sqrt{x^2+y^2}\ dz\ dy\ d\theta$$

$$=\int_{\theta=0}^{\theta=\pi}\int_{\phi=0}^{\phi=\frac{\pi}{2}}\int_{\rho=0}^{\rho=4} \rho\sin\phi\,(\rho^2\sin\phi)\,d\rho\ d\phi\ d\theta$$

$$=\int_{\theta=0}^{\theta=\pi}\int_{\phi=0}^{\phi=\frac{\pi}{2}}\int_{\rho=0}^{\rho=4} \rho^3\sin^2\phi\,d\rho\ d\phi\ d\theta$$

$$=\int_{\theta=0}^{\theta=\pi}\int_{\phi=0}^{\phi=\frac{\pi}{2}} \frac{\rho^4}{4}\sin^2\phi\Big|_{\rho=0}^{\rho=4} d\phi\ d\theta$$ (Integrate with respect to ρ)

$$=\int_{\theta=0}^{\theta=\pi}\int_{\phi=0}^{\phi=\frac{\pi}{2}} [\frac{4^4}{4}\sin^2\phi-0]d\phi\ d\theta=\int_{\theta=0}^{\theta=\pi}\int_{\phi=0}^{\phi=\frac{\pi}{2}} 64\sin^2\phi\,d\phi\ d\theta$$ (Sub in limits and simplify)

$$=\int_{\theta=0}^{\theta=\pi}\int_{\phi=0}^{\phi=\frac{\pi}{2}} 64\left[\frac{1-\cos 2\phi}{2}\right] d\phi\ d\theta$$ (Use trig identity $\sin^2 u=\frac{1-\cos 2u}{2}$)

$$=\int_{\theta=0}^{\theta=\pi}\int_{\phi=0}^{\phi=\frac{\pi}{2}} 32(1-\cos 2\phi)\,d\phi\ d\theta=\int_{\theta=0}^{\theta=\pi}\int_{\phi=0}^{\phi=\frac{\pi}{2}} (32-32\cos 2\phi)d\phi\ d\theta$$ (Simplify and dist 32)

$$=\int_{\theta=0}^{\theta=\pi} (32\phi-32(\frac{1}{2})\sin 2\phi)\Big|_{\phi=0}^{\phi=\frac{\pi}{2}}$$ (Integrate with respect to φ, use u - du sub for cos2ϕ)

$$=\int_{\theta=0}^{\theta=\pi} (32\phi-16\sin 2\phi)\Big|_{\phi=0}^{\phi=\frac{\pi}{2}} d\theta=\int_{\theta=0}^{\theta=\pi} [32\left(\frac{\pi}{2}\right)-16\sin 2(\frac{\pi}{2})]-(32(0)-16\sin 0)\,d\theta$$

$$=\int_{\theta=0}^{\theta=\pi} (16\pi-16\sin\pi-0)d\theta=\int_{\theta=0}^{\theta=\pi} (16\pi-16(0))d\theta$$

$$=\int_{\theta=0}^{\theta=\pi} 16\pi\ d\theta=16\pi\theta\Big|_{\theta=0}^{\theta=\pi}$$ (Integrate with respect to θ)

$$=16\pi(\pi)-0=\boxed{16\pi^2}$$

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