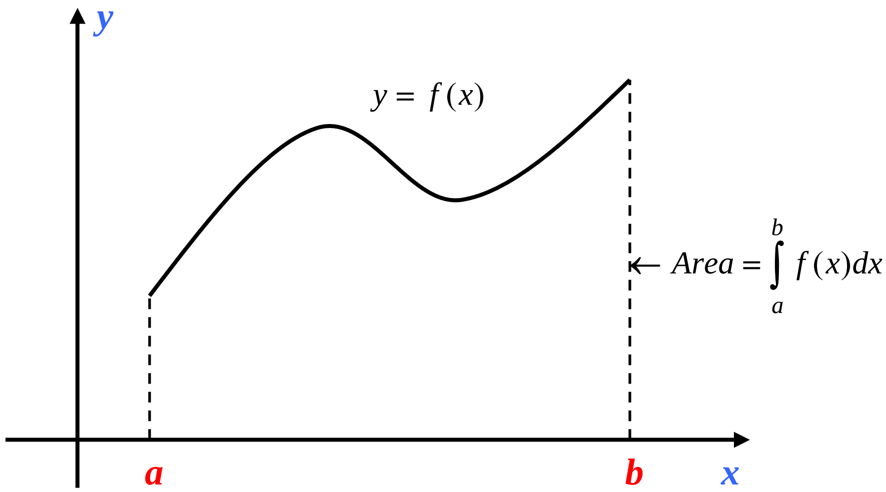
Section 13.2: Line Integrals

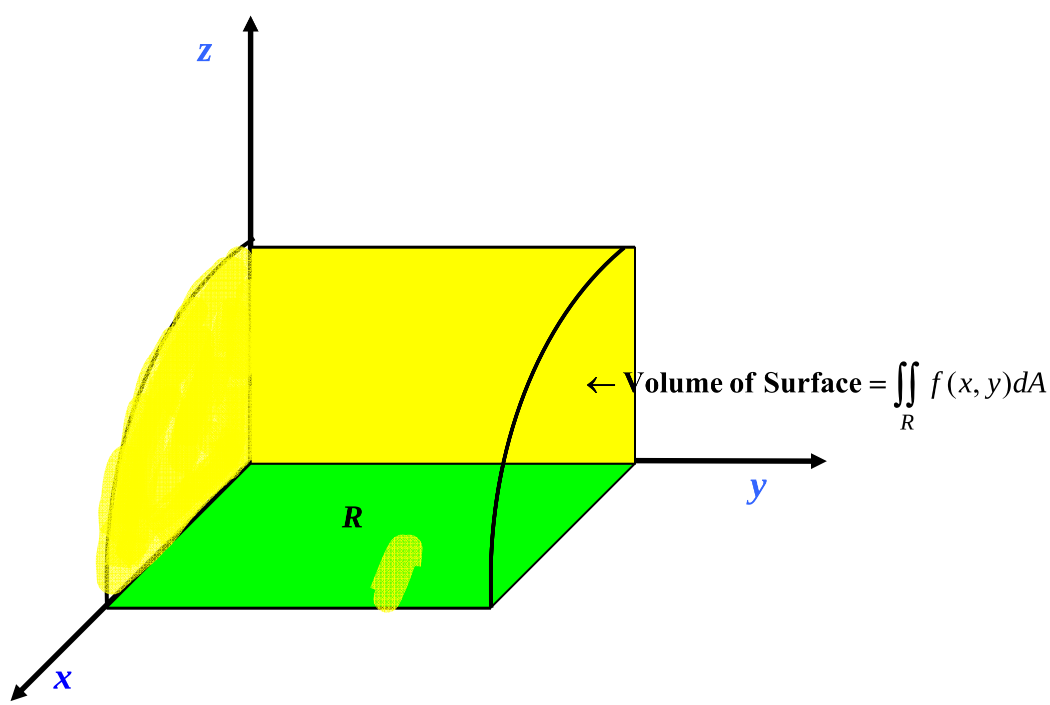
Practice HW from Stewart Textbook (not to hand in)
p. 921 # 1-13 odd, 17, 19

Line Integrals

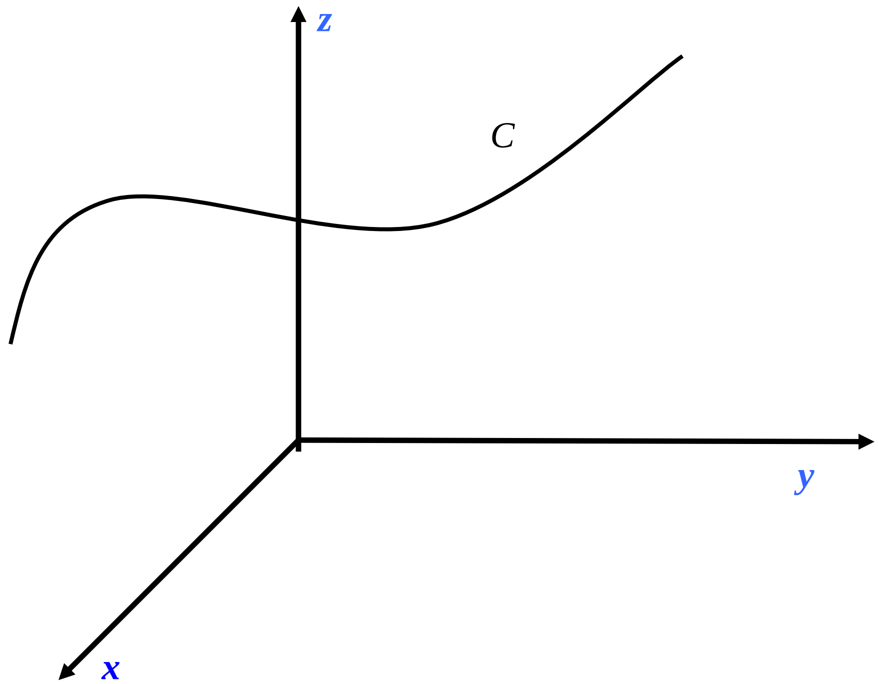
Recall that for a single integral $\int_a^b f(x)\,dx$, we integrate over a closed interval [*a*, *b*].



For double integrals, we integrate over a two dimensional region R.



For line integrals of the form $\int_C f(x, y)\, ds$, we integrate over a curve C in space.



To evaluate $\int_C f(x, y)\, ds$, a useful fact for a curve C given by the vector function $\mathbf{r}(t)$ is that $ds = |\mathbf{r}'(t)|$, we evaluate line integrals in the following manner.

Evaluating Line Integrals

Let f be a continuous function over a region containing a curve C. Then

$$2D: \begin{cases} C: r(t) = x(t)\mathbf{i} + y(t)\mathbf{j}, where \;\; a \le t \le b. \; Then \\ \int_{\mathbf{C}} f(x, y) ds = \int_{t=a}^{t=b} f(x(t), y(t)) \,|\, \mathbf{r}'(t) \,|\, dt = \int_{t=a}^{t=b} f(x(t), y(t)) \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt \end{cases}$$

$$3D: \begin{cases} C: r(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}, where \;\; a \le t \le b. \; Then \\ \int_{\mathbf{C}} f(x, y) ds = \int_{t=a}^{t=b} f(x(t), y(t), z(t)) \,|\, \mathbf{r}'(t) \,|\, dt = \int_{t=a}^{t=b} f(x(t), y(t), z(t)) \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\, dt \end{cases}$$

Example 1: Evaluate the line integral $\int_C (y/x)\,ds$ over the path $C: x = t^4, y = t^3, 1 \le t \le 2$.

Solution:

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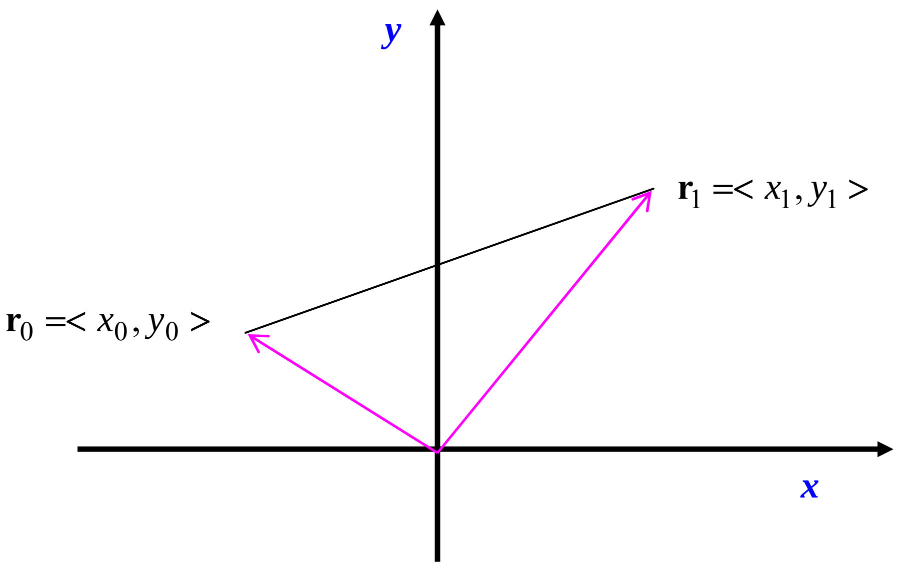
Notes

1. Sometimes the parametric equations describing the curve C are not given.

   a. For curves occurring along a circular path $x^2 + y^2 = r^2$, use the parameterization $x = r\cos t$ and $y = r\sin t$.

   b. For a straight line path occurring between the line segment that starts at the vector $\mathbf{r}_0 =< x_0, y_0 >$ and ends at vector $\mathbf{r}_1 =< x_1, y_1 >$, a parameterization is given by'

$$\mathbf{r}(t) = (1-t)\mathbf{r}_0 + t\,\mathbf{r}_1, \qquad 0 \le t \le 1$$



2. For line integrals that are defined on multiple curves that are joined, we typically calculate the line integral of each and add the results (see p. 914 Figure 4 and the discussion at the top of the page.

Example 2: Find the line integral $\int_C (x^2 + y^2)\, ds$ along the path C consisting of the upper half of the circle $x^2 + y^2 = 9$ and along the straight line segment joining the points (-3, 0) and (1, -3).

Solution:

Example 3: Determine $\int_C x^2 z\, ds$ where C is the line segment from (0, 6, -1) to (4, 1, 5).

Solution: Since this problem is defined in 3 dimensions, we must use the formula

$$\int_C f(x,y)ds = \int_{t=a}^{t=b} f(x(t),y(t),z(t))\,|\,\mathbf{r}'(t)\,|\,dt = \int_{t=a}^{t=b} f(x(t),y(t),z(t))\sqrt{[x'(t)]^2+[y'(t)]^2+[z'(t)]^2}\,dt$$

To used this formula, we must define the straight line path. We use the formula

$$\mathbf{r}(t) = (1-t)\mathbf{r}_0 + t\,\mathbf{r}_1, \qquad 0 \le t \le 1.$$

Here, we assign $\mathbf{r}_0 = <0,6,-1>$ and $\mathbf{r}_1 = <4,1,5>$. Thus

$$\begin{aligned}\mathbf{r}(t) &= (1-t)\mathbf{r}_0 + t\,\mathbf{r}_1\\ &= (1-t) <0,6,-1> + t <4,1,5>\\ &= <0, 6(1-t), -1(1-t)> + <4t, t, 5t>\\ &= <0, -6t+6, t-1> + <4t, t, 5t>\\ &= <4t, -5t+6, 6t-1>\end{aligned}$$

Thus, $\mathbf{r}(t) = <4t,-5t+6,6t-1> = 4t\,\mathbf{i} + (-5t+6)\,\mathbf{j} + (6t-1)\,\mathbf{k}$, $0 \le t \le 1$. Thus the path is

$$C: \; x(t) = 4t,\; y(t) = -5t+6,\; z(t) = 6t-1, \quad 0 \le t \le 1.$$

Since $f(x,y,z) = x^2 z$, $x'(t) = 4$, $y'(t) = -5$, and $z'(t) = 6$, we have

$$\begin{aligned}\int_C x^2 z\, ds &= \int_{t=a}^{t=b} f(x(t),y(t),z(t))\sqrt{[x'(t)]^2+[y'(t)]^2+[z'(t)]^2}\,dt\\ &= \int_{t=0}^{t=1} (4t)^2(6t-1)\sqrt{[4]^2+[-5]^2+[6]^2}\,dt\\ &= \int_{t=0}^{t=1} 16t^2(6t-1)\sqrt{16+25+36}\,dt\\ &= 16\sqrt{77}\int_{t=0}^{t=1}(6t^3-t^2)dt\\ &= 16\sqrt{77}\left(6\frac{t^4}{4} - \frac{1}{3}t^3\right)\Big|_{t=0}^{t=1}\\ &= 16\sqrt{77}\left(\frac{3}{2}(1)^4 - \frac{1}{3}(1)^3\right) = 16\sqrt{77}\left(\frac{9}{6}-\frac{2}{6}\right) = 16\sqrt{77}\left(\frac{7}{6}\right) = \boxed{\frac{56}{3}\sqrt{77}}\end{aligned}$$

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Line Integrals in Differential Form

We can define the line integral with respect to each variable separately. For a curve C in space, $a \le t \le b$, if we define $dx = x'(t)dt$, $dy = y'(t)dt$, and $dz = z'(t)dt$, we define the line integral of $f(x, y, z)$ with respect to *x*, *y*, and *z* as follows:

Line Integral with respect to *x*: $\int_C f(x, y, z)dx = \int_{t=a}^{t=b} f(x(t), y(t), z(t))\, x'(t)\, dt$

Line Integral with respect to *y*: $\int_C f(x, y, z)dx = \int_{t=a}^{t=b} f(x(t), y(t), z(t))\, y'(t)\, dt$

Line Integral with respect to z: $\int_C f(x, y, z)dx = \int_{t=a}^{t=b} f(x(t), y(t), z(t))\, z'(t)\, dt$

These line integrals can be defined together. In this case, we write

$$\int_C P(x, y, z)\, dx + \int_C Q(x, y, z)\, dy + \int_C R(x, y, z)\, dz = \int_C P(x, y, z)dx + Q(x, y, z)dy + R(x, y, z)dz$$

Example 4: Determine $\int_{\mathbf{C}} z\,dx + x\,dy + y\,dz$ for $C: x = t^2, y = t^3, z = t^2, 0 \le t \le 1$.

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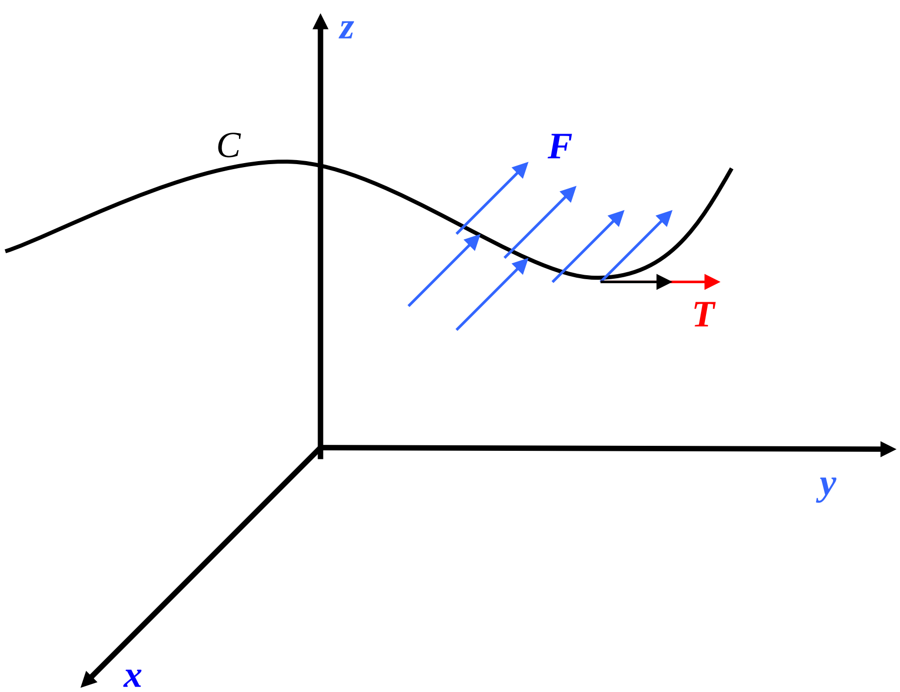
Line Integrals of Vector Fields

In physics, the work to move an object in a straight line path is the force applied to the object multiplied to the distance the object is moved. If a function $F(x)$ representing the force applied to the object at a certain position x, then the work defined on the closed interval $[a, b]$ is defined to be

$$\text{Work} = \int_a^b F(x)dx$$

Suppose we have a force field given by the vector field **F**. We can define the work defined on an object moving along a curve C as follows. Work is only done by the force in the direction of the moving object. The unit tangent vector $\mathbf{T}(t)$ points in the direction of motion. Since the force in likelihood does not in likelihood point in the exact direction of motion, we project (find the scalar projection) the vector force field **F.**



Then,

$$\text{The force projected on the tangent vector} = \frac{\mathbf{F}\cdot\mathbf{T}}{|\mathbf{T}|} = \mathbf{F}\cdot\mathbf{T}$$

Since $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$, we have

$$\mathbf{F} \cdot \mathbf{T}\, ds = \mathbf{F} \cdot \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} |\mathbf{r}'(t)|\, dt = \mathbf{F} \cdot \mathbf{r}'(t) dt = \mathbf{F} \cdot d\mathbf{r}$$

Integrating, we have the line integral for a vector field.

Line Integral For a Vector Field

Given a vector field **F**, the line integral of **F** on a curve **C** given by **r**(t) **is**

$$\int_C \mathbf{F} \cdot dr = \int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_a^b \mathbf{F}(x(t), y(t), z(t)) \cdot \mathbf{r}'(t)\, dt$$

Note: If **F** is a force field, $\int_C \mathbf{F} \cdot dr$ represents the work done by a force field.

Example 5: Evaluate $\int_C \mathbf{F} \cdot dr$ for $\mathbf{F}(x, y) = 2xy\,\mathbf{i} + (x^2 - y)\,\mathbf{j}$ .for the curve C represented by $\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j}$, $0 \le t \le \frac{\pi}{2}$

Solution:

Example 6: Evaluate $\int_C \mathbf{F} \cdot dr$ for $\mathbf{F}(x, y, z) = x^2\ \mathbf{i} + y^2\ \mathbf{j} + z^2\ \mathbf{k}$ .for the curve C represented by $\mathbf{r}(t) = t\ \mathbf{i} + t^2\ \mathbf{j} + t^3\ \mathbf{k}$, $0 \le t \le 1$

Solution: For the path $C\ : \mathbf{r}(t) = t\ \mathbf{i} + t^2\ \mathbf{j} + t^3\ \mathbf{k}$, we have $x(t) = t, y(t) = t^2$, and $z(t) = t^3$. Thus,

$$\mathbf{F}(x(t), y(t), z(t)) = \mathbf{F}(t, t^2, t^3) = t^2\ \mathbf{i} + (t^2)^2\ \mathbf{j} + (t^3)^2\ \mathbf{k} = t^2\ \mathbf{i} + t^4\ \mathbf{j} + t^6\ \mathbf{k} = < t^2, t^4, t^6 > .$$

Since $\mathbf{r}'(t) = \mathbf{i} + 2t\ \mathbf{j} + 3t^2\ \mathbf{k} = < 1, 2t, 3t^2 >$, we have

$$\int_C \mathbf{F} \cdot dr = \int_{t=a}^{t=b} \mathbf{F}(x(t), y(t)) \cdot \mathbf{r}'(t) dt$$

$$= \int_{t=0}^{t=1} < t^2, t^4, t^6 > \cdot < 1, 2t, 3t^2 > dt$$

$$= \int_{t=0}^{t=1} [(t^2)(1) + (t^4)(2t) + (t^6)(3t^2)] dt$$

$$= \int_{t=0}^{t=1} (t^2 + 2t^5 + 3t^8)\ dt$$

$$= (\frac{1}{3}t^3 + \frac{1}{3}t^6 + \frac{1}{3}t^9 \Big|_{t=0}^{t=1}$$

$$= (\frac{1}{3}(1)^3 + \frac{1}{3}(1)^6 + \frac{1}{3}(1)^9) - 0$$

$$= \frac{1}{3} + \frac{1}{3} + \frac{1}{3}$$

$$\boxed{= 1}$$

$\blacksquare$