Section 13.3: The Fundamental Theorem of Line Integrals

Practice HW from Stewart Textbook (not to hand in)
p. 931 # 3-9 odd, 13-21 odd

Conservative Vector Fields

A vector field **F** is called conservative if there exists a differentiable function $f(x, y)$ where

$$\nabla f(x, y) = \mathbf{F}$$

Example 1: Given the function $f(x, y, z) = x^2 y - yz + xyz$, find the conservative vector field **F.**

Solution:

Test for Conservative Vector Fields in the 2D Plane

The vector field $\mathbf{F}(x, y) = P\,\mathbf{i} + Q\,\mathbf{j}$ is conservative if and only if

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$$

Example 2: Determine whether or not $\mathbf{F}(x, y) = \frac{1}{y}\mathbf{i} - \frac{2x}{y^2}\mathbf{j}$ is a conservative. If it is, find a function f where $\nabla f = \mathbf{F}$.

Solution:

Example 3: Determine whether or not $\mathbf{F}(x, y) = 2xy\,\mathbf{i} + (x^2 - y)\,\mathbf{j}$ is a conservative. If it is, find a function f where $\nabla f = \mathbf{F}$.

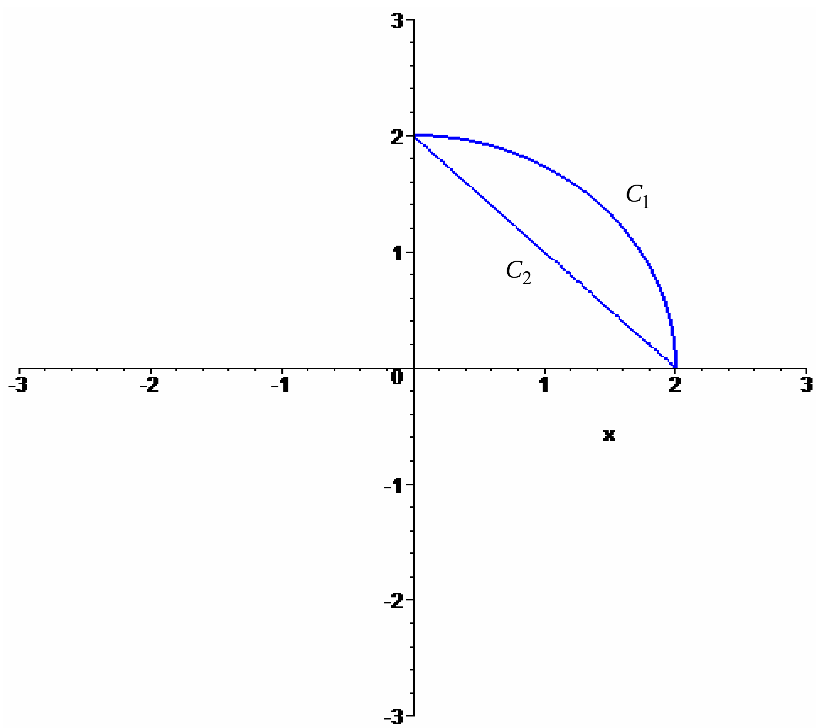
Solution:

The Fundamental Theorem of Line Integrals

Consider the consider the vector field $\mathbf{F}(x, y) = 2xy\,\mathbf{i} + (x^2 - y)\,\mathbf{j}$ and suppose we evaluate $\int_C \mathbf{F} \cdot dr$ over the two paths

$C_1 : \mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j},\ 0 \le t \le \frac{\pi}{2}$ and $C_2 : \mathbf{r}(t) = (2 - 2t)\,\mathbf{i} + 2t\,\mathbf{j}, 0 \le t \le 1$.

The paths can be seen in the following diagram:



For the path $C_1 : \mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j},\ 0 \le t \le \frac{\pi}{2}$, we saw from Example 5 in the Section 11.2 notes that $\int_{C_1} \mathbf{F} \cdot dr = -2$

For the path $C_2 : \mathbf{r}(t) = (2 - 2t)\,\mathbf{i} + 2t\,\mathbf{j}$, we have $x(t) = 2 - 2t, y(t) = 2t$. Thus,

$$\begin{aligned}\mathbf{F}(x(t), y(t)) = \mathbf{F}(2-2t, 2t) \\ &= 2(2-2t)(2t)\,\mathbf{i} + ((2-2t)^2 - 2t)\,\mathbf{j} \\ &= (8t - 8t^2)\,\mathbf{i} + (4 - 4t - 4t + 4t^2 - 2t)\,\mathbf{j} \\ &= (8t - 8t^2)\,\mathbf{i} + (4 - 10t + 4t^2)\,\mathbf{j}\end{aligned}.$$

Since $\mathbf{r}'(t) = -2\,\mathbf{i} + 2\,\mathbf{j}$, we have

$$\int_{C_2} \mathbf{F} \cdot dr = \int_{t=0}^{t=1} \mathbf{F}(x(t), y(t)) \cdot \mathbf{r}'(t) dt$$
$$= \int_{t=0}^{t=1} < 8t - 8t^2, 4 - 10t + 4t^2 > \cdot < -2, 2 > dt$$
$$= \int_{t=0}^{t=1} [(8t - 8t^2)(-2) + (4 - 10t + 4t^2)(2)]\, dt$$
$$= \int_{t=0}^{t=1} (-16t + 16t^2 + 8 - 20t + 8t^2)\, dt$$
$$= \int_{t=0}^{t=1} (24t^2 - 36t + 8)\, dt$$
$$= (8t^3 - 18t^2 + 8t)\Big|_{t=0}^{t=1}$$
$$= (8(1)^3 - 18(1)^2 + 8(1)) - 0$$
$$= -2$$

Note that for the paths C_1 and C_2, the initial point and ending points are the same. That is each graph starts at the point (2, 0) ($t = 0$ for both C_1 and C_2) and both end at the point (0, 2) ($t = \frac{\pi}{2}$ for C_1 and $t = 1$ for C_2). The line integrals for both of these paths are

$$\int_{C_1} \mathbf{F} \cdot dr = -2 = \int_{C_2} \mathbf{F} \cdot dr$$

For <u>conservative</u> <u>vector</u> <u>fields</u>, this common values for the line integrals is not a coincidence.

Fact: The value of $\int_C \mathbf{F} \cdot dr$ is independent of path if and only if the vector field **F** is conservative.

If we know the function f where $\nabla f(x, y) = \mathbf{F}$, there is a much easier way to evaluate to line integral of a conservative vector field **F**, which we state next.

Fundamental Theorem of Line Integrals

Given a path $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j},\ a \le t \le b$, with initial point $(x(a), y(a))$ and terminal point $(x(b), y(b))$. Then if $\mathbf{F}$ is a <u>conservative</u> vector field with function f where $\nabla f(x, y) = \mathbf{F}$, then

$$\int_C \mathbf{F} \cdot dr = f(x(b), y(b)) - f(x(a), x(a))$$

The function has a natural extension of functions of 3 variables.

Example 4: Use to fundamental theorem of line integrals to evaluate $\int_C \mathbf{F} \cdot dr$ for $\mathbf{F}(x, y) = 2xy\,\mathbf{i} + (x^2 - y)\,\mathbf{j}$ over the path C from the point (2, 0) to (0, 2).

Solution:

Example 5: Evaluate $\int_C \mathbf{F} \cdot dr$ for $\mathbf{F}(x, y, z) = x^2\ \mathbf{i} + y^2\ \mathbf{j} + z^2\ \mathbf{k}$ over the path $C : \mathbf{r}(t) = \sin t\ \mathbf{i} + \cos t\ \mathbf{j} + t^2\ \mathbf{k},\ 0 \le t \le \dfrac{\pi}{2}$.

Solution:

$\blacksquare$

Note: If a curve C given by $\mathbf{r}(t)$ is closed, that is, $\mathbf{r}(a) = \mathbf{r}(b)$ for $a \le t \le b$ where the path has the same initial and terminal points, then $\int_C \mathbf{F} \cdot dr = 0$ if $\mathbf{F}$ is conservative.