Mathematical Modeling

I. UNIT OVERVIEW & PURPOSE:

Students will gain a deeper understanding of the use of polynomial, exponential, and logarithmic functions by applying them to real-world situations including analyzing factors that contribute to automobile accidents in Virginia caused by deer, analyzing road speed and fuel economy, and analyzing the affects of raising the US debt ceiling. They will understand the meaning of the important features of the graphs of these functions, i.e. the intercepts, maximum and/or minimum points, and the asymptotes, and make interpretations in the context of the problems.

II. UNIT AUTHOR:

Cynthia Cowley. Piedmont Governor's School.

III. COURSE:

Mathematical Modeling: Capstone Course

IV. CONTENT STRAND:

Data Analysis and Probability

V. OBJECTIVES:

Students will:

- Plot data points on graph paper to determine the type of relation (polynomial, exponential, logarithmic);
- Determine the line or curve of best fit by hand;
- Enter data into List function of graphing calculator and calculate the curve of best fit using the graphing calculator;
- Use Excel to graph data and determine the curve of best fit, i.e. trend (linest), exponential (growth), or logarithmic (log);
- Analyze the correlation coefficient to determine if the curve is a good model of the data;
- Verbalize the meaning of the pertinent information of the curve, i.e. intercepts, maximum and/or minimum points, and asymptotes in the context of the problem;
- Prepare a PowerPoint presentation to illustrate the problem, the calculations, and the discussion regarding the model found for the problem.

VI. MATHEMATICS PERFORMANCE EXPECTATION(S):

MPE.2 Collect and analyze data, determine the equation of the curve of best fit, make predictions, and solve real-world problems, using mathematical models.

Mathematical models will include polynomial, exponential, and logarithmic functions.

VII. CONTENT:

This unit will provide students with an understanding and appreciation of polynomial, exponential, and logarithmic functions by modeling real-world situations to which they can relate. Students will learn how to analyze data and create a model for the data both by hand and by using a graphing calculator. Students will also determine the limitations (if any) of such models and why they exist.

VIII. REFERENCE/RESOURCE MATERIALS:

Students will need access to a computer lab with internet access for the purpose of collecting data and completing research. A classroom set of TI-84+ graphing calculators will also be provided.

IX. PRIMARY ASSESSMENT STRATEGIES:

Students will be assessed on their process of completing the lessons including the correctness of the mathematical computations, the ability to discuss and describe the significance and the meaning of the models they calculate, and the preparation and presentation of the calculations and their significance.

X. EVALUATION CRITERIA:

Two grading rubrics are provided with lesson 1. One will be used to assess the mathematical computations and discussion and the second will be used to assess the PowerPoint and presentation. Since lesson 2 and lesson 3 are similar in nature, the same grading rubrics will be used for all lessons.

XI. INSTRUCTIONAL TIME:

Three lessons are included which are similar in nature. Teachers may wish to offer students an option of choosing one of the three to complete, or completing all three. Approximately two weeks based on a 90-minute class period would be needed if all three lessons are to be completed.

Lesson 1

Strand

Data Analysis and Probability

Mathematical Objectives

Curve of best fit. In this lesson students will collect and analyze data (websites provided under Materials and Resources) to determine if there is a correlation between the deer population in

Virginia in the years 1990 – 2009 and the number of big game licenses purchased during that same time period. They will use their knowledge of polynomial, exponential, and logarithmic functions to create a mathematical model that will represent the relationship between these two variables. They will list simplifying assumptions, discuss the validity of their model by examining the correlation coefficient and discuss how the model changes if assumptions change.

Mathematics Performance Expectations

2. The student will collect and analyze data, determine the equation of the curve of best fit, make predictions, and solve real-world problems, using mathematical models. Mathematical models will include polynomial, exponential, and logarithmic functions.

Specifically, the student will:

- Recognize the general shape of a function (polynomial, exponential, and logarithmic)
- Use graphing calculator to investigate the shapes and behaviors of these functions
- Write an equation given the graph of a function
- Analyze functions to find the real-world meaning of the x- and y-intercepts, local and absolute maxima and minima, and asymptotes
- Find the value of a function for an element in its domain

Related SOL

- A.2c (perform operations on polynomials including factoring)
- A.7b,c,d (investigate and analyze function families and their characteristics both analytically and graphically including domain and range, zeros, x- and y-intercepts)
- A.11 (determine the equation of the curve of best fit)
- AII/T.8 (investigate and describe the relationships among solutions of an equation,
- zeros of a function, *x*-intercepts of a graph, and factors of a polynomial expression)
- AII/T.9 (collect and analyze data, determine the equation of the curve of best fit, make
- predictions, and solve real-world problems, using mathematical models. Mathematical models will include polynomial, exponential, and logarithmic functions)
- AFDA.1 (investigate and analyze linear, quadratic, exponential, and logarithmic families and their characteristics)
- AFDA.2 (write an equation, given the graph of a linear, quadratic, exponential, or logarithmic function)
- AFDA.3 (collect data and generate an equation for the curve of best fit to model real-world problems or applications)

NCTM Standards

- generalize patterns using explicitly defined and recursively defined functions;
- understand relations and functions and select, convert flexibly among, and use various representations for them;
- analyze functions of one variable by investigating rates of change, intercepts, zeros, asymptotes, and local and global behavior;
- use symbolic algebra to represent and explain mathematical relationships
- identify essential quantitative relationships in a situation and determine the class or classes of functions that might model the relationships;

- use symbolic expressions, including iterative and recursive forms, to represent relationships arising from various contexts;
- draw reasonable conclusions about a situation being modeled.

Additional Objectives for Student Learning
Students will
- Use internet searches and/or websites provided to gain information regarding the significance of the problem of focus and to retrieve data
- Use technology including TI-84+ graphing calculators and Microsoft Excel to find the curve of best fit for a set of data
- Use Microsoft PowerPoint to create a presentation

Materials/Resources
Classroom set of TI-84+ Graphing Calculators
Access to computer lab with internet access
http://www.hunting-fishing-virginia.com/blog/hunting/virginia-whitetail-deer-population.html
http://www.dgif.virginia.gov/wildlife/deer/harvest/index.asp
http://www.census.gov/prod/www/abs/fishing.html (see Table 2)
http://www.areaconnect.com/population.htm?s=VA
SmartBoard with TI Smartview Calculator or Emulator
Graph paper
Paper

Assumption of Prior Knowledge
- The typical student would have successfully completed algebra 1, algebra 2, geometry, and algebra functions with data analysis. Students should be able to write the equation of a line using two points on the line, factor polynomials, convert between graphic and symbolic forms of functions, and be able to determine the equation of the curve of best fit of a set of data, make predictions, and solve real-world problems, using mathematical models including polynomial, exponential, and logarithmic functions.
- To be successful with this lesson a typical student should be operating on level 2 of the Van Hiele scale – abstraction. Students have a good understanding of properties and understand that one set of properties may imply another property.
- Students may find it difficult to develop models by hand for polynomial functions of degree greater than one, and may find it difficult to develop models for data fitting an exponential or logarithmic pattern.

Concepts relevant to this unit that should be covered prior to this unit include:
- an investigation of linear, quadratic, exponential, and logarithmic families algebraically and graphically
- writing an equation given the graph of a function (linear, quadratic, exponential, and logarithmic)
- an investigation and description of the relationships among solutions of an equation, zeros of a function, x-intercepts of a graph, and factors of a polynomial expression
- an ability to recognize the general shape of a function (polynomial, exponential, and logarithmic)

Introduction: Setting Up the Mathematical Task

In this lesson, students will find a function that models a set of data and use the model to interpolate and extrapolate, i.e. make predictions of the data.

Youtube video: (3 minutes)
http://www.hunting-fishing-virginia.com/blog/hunting/virginia-whitetail-deer-population.html

According to the National Highway Traffic Safety Administration, Virginia was ranked in the top ten worst states for deer collisions in a study conducted in 2004-2005. One factor that helps to control the population of deer is the number of big game hunters. Examine the deer kill and the number of big game hunters in Virginia during the years 1991- 2006. (Data for number of big game hunters is only available on 5-year increments). Determine the type of relationship (polynomial, exponential, or logarithmic) that exists for number of big game hunters over time and the deer kill over time.

- State your simplifying assumptions. (small groups 10 minutes followed by whole class 5 minutes)
- Identify dependent and independent variables in your mathematical model. (small groups 5 minutes followed by whole class 2 minutes)
- Identify other information necessary (to be gathered using technology) in order to solve the problem.

The teacher will facilitate the activity by observing and answering questions when needed. A discussion including some of the possible simplifying assumptions would be helpful. These may include:

1. Coyotes are not helping to control the deer population.
2. Hunters only have on average one day per week to hunt which results in approximately 4 days.
3. License fees have remained constant during this period.
4. The reproduction rates of deer will follow a normal pattern, i.e. there will not be a surge of twins and triplets other than what is normal.
5. Hunters will shoot either sex deer.
6. Using census data for Virginia in 1991 and 2001, the percentage of big game hunters in Virginia is decreasing.

Students will be paired in groups of two based on interests and ability levels as determined previously by teacher through surveys, observations, and assessments. Heterogeneous grouping would be advised for the first of the three lessons, with a possible switch to homogenous grouping for the second and third lessons to be determined by the teacher.

Student Exploration 1:

Students will collect data on the deer kill in Virginia and the number of big game hunters in Virginia during the time period. They will use prior knowledge of plotting points, estimation, and calculating the regression equation using a graphing utility to find a model for each set of data that was collected. To develop a relationship between the two sets of data over time, the

deer kill function $d(x)$ will be divided by the number of big game hunters $h(x)$ where x represents time with the year 2000 as $x = 0$.

Student/Teacher Actions:

To explore and develop this model the following are important points of discussion:

- Students will enter each set of data into the Lists menu of the graphing calculator where L_1 is time (x), L_2 is deer kill, and L_3 is number of big game hunters.
- Students will adjust the viewing window of the calculator to accommodate the data that was entered
- Using the statplots menu, students will plot the data on the graphing calculator, one set at a time i.e. L_1, L_2 and then L_1, L_3.
- Determine if the points appear to follow a linear, quadratic, exponential, or logarithmic regression
- Use the graphing calculator to find the regression equation for each set of data
- Using the diagnostics option on the graphing calculator, determine the correlation coefficient for the regression equation that you found
- Use Excel to graph data and determine the curve of best fit, i.e. trend (linest), exponential (growth), or logarithmic (log)
- Use the CORREL function in Excel to find the correlation coefficient
- Based on the correlation coefficient, discuss the validity of the regression equation
- Find a model to represent deer kill vs. number of big game hunters $P(x) = \frac{d(x)}{h(x)}$
- Discuss the affect of big game hunters on the deer population.

Assessment

1. Using the appropriate model, students will determine what the deer kill in Virginia will be for the years
   a) 2012
   b) 2016
   c) 2020
2. In what year will the deer kill be 275,000?
3. Using the appropriate model, students will determine the number of big game hunters in Virginia for the years
   a) 2012
   b) 2016
   c) 2020.
4. Suppose Sunday hunting was instituted in Virginia in the year 2015. What might you expect to change in your model?

Extensions and Connections (for all students)

To extend the material for students ready to move forward have them find models for the same situation in Pennsylvania (the state ranked the worst for auto accidents involving deer).

Strategies for Differentiation

Use different tasks to find the regression equation. Students grouped in pairs according to ability levels previously determined.

- Task 1: Plot the data on graph paper using a separate graph for time vs. big game hunters and a separate graph for time vs. deer kill. Use time as the independent variable and the number of big game hunters or deer kill as the dependent variable. Estimate the regression equation using the "eyeball" method, draw the curve, and use the graphs to determine the number of big game hunters and number of deer killed in the year 2011.
- Task 2: Plot the data on graph paper as described above. Estimate the regression equation using the "eyeball" method, and draw the curve. Identify two points on the curve and use the points to write the equation of the curve. Use the equation to determine the number of big game hunters and number of deer killed in the year 2011.
- Task 3: Enter the data into the Lists menu of the graphing calculator using time as the independent variable and the number of big game hunters or deer kill as the dependent variable. Use the stat plots menu to graph the points. Calculate the regression equation. Transfer this equation to the $y =$ menu and graph, adjusting the window to view the curve. Find the value of y (the number of big game hunters or deer kill) when $x = 11$ (time). Find the value of the correlation coefficient and discuss its significance and affect on the prediction.
- Task 4: Enter the data into an Excel spreadsheet using time as the independent variable and the number of big game hunters or deer kill as the dependent variable. Graph the data by inserting a scatter plot. After observing the results of the graph, determine whether to use trend, exponential, or a logarithmic function to represent the data. Find the value of y (the number of big game hunters or deer kill) when $x = 11$ (time). Find the value of the correlation coefficient using CORREL and discuss its significance and affect on the prediction.

Student Exploration 2

Individual work

Students should research the deer population in Virginia and any factors having an effect on the population. This may include predators, disease, hunting, car accidents, etc. Use internet search engines to find information. Make a list of any websites from which you gather information.

Small Group Work

- Students will share information found individually and come to a consensus on the factors affecting the deer population, listing them in order from greatest affect to least affect.
- Students will prepare a PowerPoint presentation to include all mathematical computations calculated in Student Exploration 1, beginning with raw data, graphs, modeling functions, etc. The PowerPoint should also include their research as stated above.
- Requirements for PowerPoint:
    - A minimum of 10 slides
    - Use of graphics to display data
    - Calculations typed using the equation editor
    - Use of a common theme and design
    - A list of sources

Rubric to Assess Mathematical Content and Discussion

Objective	Beginning	Developing	Accomplished
Plot data points on graph paper to determine the type of relation	The student does not use an appropriate and/or consistent scale on graph 2 points	The student uses an appropriate scale but plots one or more points incorrectly. 3-6 points	The student uses an appropriate scale, plots all points correctly, and determines the type of relation. 7-10 points
Determine the line or curve of best fit by hand	The student draws a line or curve that best fits the data. 2 points	The student draws a line or curve that best fits the data, but has errors in calculation. 3-6 points	The student draws a line or curve that best fits the data and correctly calculates the function. 7-10 points
Enter data into List function of graphing calculator and calculate the curve of best fit.	The student correctly enters the data into List function of graphing calculator. 2 points	The student correctly enters the data into List function of graphing calculator but chooses the wrong type of regression. 3-6 points	The student correctly enters the data into List function of graphing calculator and correctly calculates the curve of best fit. 7-10 points
Use Excel to graph data and determine the curve of best fit, i.e. trend (linest), exponential (growth), or logarithmic (log)	The student correctly enters the data into Excel and creates a scatter plot of the data. 2 points	The student correctly enters the data into Excel, creates a scatter plot, but chooses the wrong type of regression. 3-6 points	The student correctly enters the data into Excel, creates a scatter plot, chooses the proper regression, and uses it to make predictions. 7-10 points
Analyze the correlation coefficient to determine if the curve is a good model of the data	The student correctly finds the correlation coefficient. 2 points	The student correctly finds the correlation coefficient but incorrectly interprets it. 3-6 points	The student correctly finds the correlation coefficient and correctly interprets its meaning. 7-10 points
Verbalize the meaning of the pertinent information of the curve, i.e.	The student verbalizes the meaning of the y-intercept.	The student verbalizes the meaning of both intercepts.	The student verbalizes the meaning of all information

intercepts, maximum and/or minimum points, and asymptotes in the context of the problem.	2 points	3-6 points	pertinent to the curve in the context of the problem. 7-10 points

Rubric to Assess PowerPoint and Presentation

Category	Fair/Poor 2 points	Good 3-5 points	Excellent 6-9 points	Superior 10 points
Originality	Presentation shows very little attempt at original thought.	Presentation shows an attempt at originality and creativity on a few slides.	Presentation shows some originality and creativity. The content is presented in an interesting way.	Presentation shows considerable originality and creativity. The content is presented in a unique and interesting way.
Sequencing of information	There is no clear plan for the organization of information.	Some information is logically sequenced.	Most information is organized in a clear, logical way. At least one item of information seems out of place.	Information is organized in a clear, logical way.
Content-accuracy	Content is confusing or contains more than one error.	The content is generally accurate, but some pieces of inaccurate.	Most of the content is accurate but there is one piece of information that might be inaccurate.	All content throughout the presentation is accurate. There are no errors.
Requirements	Two or more requirements were not completely met.	One requirement was not completely met.	All requirements were met.	All requirements were met and exceeded.
Use of graphics	Several graphics are unattractive and detract from the content of the presentation.	All graphics are attractive but a few do not seem to support the content of the presentation.	A few graphics are not attractive but all support the content of the presentation.	All graphics are attractive. Color was used for emphasis.
Organization	There was no real organization of	Content is somewhat	Headings were used to organize the	Headings and bullets were used

	the material.	logically organized.	material.	to organize the material.
Sources	Little source information was provided.	Source information was provided but incorrectly documented.	Source information was provided and some was given in desired format.	Source information was provided and correctly documented.
Execution	Presenter did not know his material.	Presenter was not prepared. He read slides and had little eye contact.	Presenter was prepared and gave an above average presentation.	Presenter was fully prepared and had good eye contact and flow.

Lesson 2

Strand
Data Analysis and Probability

Mathematical Objectives
Curve of best fit. In this lesson students will collect and analyze data (websites provided under Materials and Resources) to determine the optimum speed limit for interstate travel to obtain maximum gas mileage. They will use their knowledge of polynomial, exponential, and logarithmic functions to create a mathematical model that will represent the relationship between these two variables. They will list simplifying assumptions, discuss the validity of their model by examining the correlation coefficient and discuss how the model changes if assumptions change.

Mathematics Performance Expectations

2. The student will collect and analyze data, determine the equation of the curve of best fit, make predictions, and solve real-world problems, using mathematical models. Mathematical models will include polynomial, exponential, and logarithmic functions.

Specifically, the student will:

- Recognize the general shape of a function (polynomial, exponential, and logarithmic)
- Use graphing calculator to investigate the shapes and behaviors of these functions
- Write an equation given the graph of a function
- Analyze functions to find the real-world meaning of the x- and y-intercepts, local and absolute maxima and minima, and asymptotes
- Find the value of a function for an element in its domain

Related SOL

- A.2c (perform operations on polynomials including factoring)

- A.7b,c,d (investigate and analyze function families and their characteristics both analytically and graphically including domain and range, zeros, x- and y-intercepts)
- A.11 (determine the equation of the curve of best fit)
- AII/T.8 (investigate and describe the relationships among solutions of an equation,
- zeros of a function, *x*-intercepts of a graph, and factors of a polynomial expression)
- AII/T.9 (collect and analyze data, determine the equation of the curve of best fit, make
- predictions, and solve real-world problems, using mathematical models. Mathematical models will include polynomial, exponential, and logarithmic functions)
- AFDA.1 (investigate and analyze linear, quadratic, exponential, and logarithmic families and their characteristics)
- AFDA.2 (write an equation, given the graph of a linear, quadratic, exponential, or logarithmic function)
- AFDA.3 (collect data and generate an equation for the curve of best fit to model real-world problems or applications)

NCTM Standards

- generalize patterns using explicitly defined and recursively defined functions;
- understand relations and functions and select, convert flexibly among, and use various representations for them;
- analyze functions of one variable by investigating rates of change, intercepts, zeros, asymptotes, and local and global behavior;
- use symbolic algebra to represent and explain mathematical relationships
- identify essential quantitative relationships in a situation and determine the class or classes of functions that might model the relationships;
- use symbolic expressions, including iterative and recursive forms, to represent relationships arising from various contexts;
- draw reasonable conclusions about a situation being modeled.

Additional Objectives for Student Learning

Students will

- Use internet searches and/or websites provided to gain information regarding the significance of the problem of focus and to retrieve data
- Use technology including TI-84+ graphing calculators and Microsoft Excel to find the curve of best fit for a set of data

Materials/Resources

Classroom set of TI-84+ Graphing Calculators
Access to computer lab with internet access
Possible websites to gather data and information
http://www.metrompg.com/posts/speed-vs-mpg.htm
*http://cta.ornl.gov/data/tedb29/Edition29_Chapter04.pdf
http://www.fueleconomy.gov/feg/drivehabits.shtml
SmartBoard with TI Smartview Calculator or Emulator
Graph paper
Paper

Assumption of Prior Knowledge

- The typical student would have successfully completed algebra 1, algebra 2, geometry, and algebra functions with data analysis. Students should be able to write the equation of a line using two points on the line, factor polynomials, convert between graphic and symbolic forms of functions, be able to find the average of a set of data, be able to find the maximum or minimum of a function, and be able to determine the equation of the curve of best fit of a set of data, make predictions, and solve real-world problems, using mathematical models including polynomial, exponential, and logarithmic functions.
- To be successful with this lesson a typical student should be operating on level 2 of the Van Hiele scale – abstraction. Students have a good understanding of properties and understand that one set of properties may imply another property.
- Students may find it difficult to develop models by hand for polynomial functions of degree greater than one, and may find it difficult to develop models for data fitting an exponential or logarithmic pattern.

Concepts relevant to this unit that should be covered prior to this unit include:

- an investigation of linear, quadratic, exponential, and logarithmic families algebraically and graphically
- writing an equation given the graph of a function (linear, quadratic, exponential, and logarithmic)
- an investigation and description of the relationships among solutions of an equation, zeros of a function, x-intercepts of a graph, and factors of a polynomial expression
- an ability to recognize the general shape of a function (polynomial, exponential, and logarithmic)

Introduction: Setting Up the Mathematical Task

In this lesson, students will find a function that models a set of data and use the model to interpolate and extrapolate, i.e. make predictions of the data.

The primary focus of fuel conservation is typically on the fuel economy of new cars and light trucks, but it is widely recognized that vehicle owners can take steps to maximize fuel economy of their vehicle in-use. Setting the maximum speed limit on interstates is often an issue that arises in times of higher gas prices. Collect and analyze data to determine the optimum speed to maximize fuel economy on interstate driving.

- State your simplifying assumptions. (small groups 10 – 15 minutes followed by whole class 5 minutes)
- Identify dependent and independent variables in your mathematical model. (small groups 5 minutes followed by whole class 2 minutes)
- Identify other information necessary (to be gathered using technology) in order to solve the problem.

The teacher will facilitate the activity by observing and answering questions when needed. A discussion including some of the possible simplifying assumptions would be helpful. These may include:

1. Vehicles considered will be the most popular passenger cars used in America in 1997*.
2. Motorists will be responsible drivers, not of the aggressive nature.
3. Vehicle owners have routine maintenance performed on their vehicles.
4. Vehicle owners use tires that have low rolling resistance, thus helping to improve gas mileage.
5. Vehicle owners use fuel efficient lubricants.

Students will be paired in groups of three based on interests and ability levels as determined previously by teacher through surveys, observations, and assessments.

Student Exploration 1:

Students will collect data on maximum fuel efficiency for the most popular U.S. models in 1997*. They will use prior knowledge of plotting points, estimation, and calculating the regression equation using a graphing utility to find a model for the data that was collected.

Student/Teacher Actions:

To explore and develop this model the following are important points of discussion:

- Students will research the fuel efficiency of each of the models in the 1997 study. Data needed includes the mpg ratio at speeds ranging from 30 to 75 mph.
- Students will find the average fuel efficiency of the models selected.
- Students will enter each set of data into the Lists menu of the graphing calculator where L_1 is speed (x), and L_2 is average fuel efficiency (y).
- Students will adjust the viewing window of the calculator to accommodate the data that was entered
- Using the statplots menu, students will plot the data on the graphing calculator.
- Determine if the points appear to follow a linear, quadratic, exponential, or logarithmic regression
- Use the graphing calculator to find the regression equation for the data
- Using the diagnostics option on the graphing calculator, determine the correlation coefficient for the regression equation that you found
- Use Excel to graph data and determine the curve of best fit, i.e. trend (linest), exponential (growth), or logarithmic (log).
- Use the CORREL function in Excel to find the correlation coefficient.
- Compare graph and answers to those found with the graphing calculator.
- Based on the correlation coefficient, discuss the validity of the regression equation
- What are the limitations of the model?
- Discuss the affect of speed on average fuel efficiency.

Assessment

- Using the appropriate model, students will determine what the fuel economy for speeds
  a) 62 miles per hour
  b) 82 miles per hour
  c) 85 miles per hour

- For what speed will the gas mileage be 32 mpg?

Extensions and Connections (for all students)

To extend the material for students ready to move forward have them gather data needed to update the study completed in 1997.

Strategies for Differentiation

Use different tasks to find the regression equation. Students grouped in pairs according to ability levels previously determined.

- Task 1: Plot the data on graph paper using the speed as the independent variable and the average fuel efficiency as the dependent variable, estimate the regression equation using the "eyeball" method, draw the curve, and use the graph to determine the maximum fuel economy.
- Task 2: Plot the data on graph paper using the speed as the independent variable and the average fuel efficiency as the dependent variable, estimate the regression equation using the "eyeball" method, and draw the curve. Identify two points on the curve, set up a system of equations to find the equation of the curve. Use the equation to determine the maximum fuel economy.
- Task 3: Enter the data into the Lists menu of the graphing calculator using the speed as the independent variable and the average fuel efficiency as the dependent variable. Use the stat plots menu to graph the points. Calculate the regression equation. Transfer this equation to the $y =$ menu and graph, adjusting the window to view the curve. Calculate the maximum point on the curve using the calc menu. Find the value of y (the average fuel efficiency) of the maximum point. Note that x represents the speed that yields the average fuel efficiency. Find the value of the correlation coefficient and discuss its significance and affect on the prediction.
- Task 4: Enter the data into an Excel spreadsheet using speed as the independent variable and the average fuel efficiency as the dependent variable. Graph the data by inserting a scatter plot. After observing the results of the graph, determine whether to use trend, exponential, or a logarithmic function to represent the data. Find the value of y (the average fuel efficiency) at its maximum point. Find the value of the correlation coefficient using CORREL and discuss its significance and affect on the prediction.

Lesson 3

Strand

Data Analysis and Probability

Mathematical Objectives

Curve of best fit. In this lesson students will collect and analyze data (websites provided under Materials and Resources) to determine a function to represent the national debt. They will use their knowledge of polynomial, exponential, and logarithmic functions to create a mathematical model that will represent the relationship between these two variables. They will list simplifying assumptions, discuss the validity of their model by examining the correlation coefficient and discuss how the model changes if assumptions change.

Mathematics Performance Expectations

2. The student will collect and analyze data, determine the equation of the curve of best fit, make predictions, and solve real-world problems, using mathematical models. Mathematical models will include polynomial, exponential, and logarithmic functions.

Specifically, the student will:

- Recognize the general shape of a function (polynomial, exponential, and logarithmic)
- Use graphing calculator to investigate the shapes and behaviors of these functions
- Write an equation given the graph of a function
- Analyze functions to find the real-world meaning of the x- and y-intercepts, local and absolute maxima and minima, and asymptotes
- Find the value of a function for an element in its domain

Related SOL

- A.2c (perform operations on polynomials including factoring)
- A.7b,c,d (investigate and analyze function families and their characteristics both analytically and graphically including domain and range, zeros, x- and y-intercepts)
- A.11 (determine the equation of the curve of best fit)
- AII/T.8 (investigate and describe the relationships among solutions of an equation,
- zeros of a function, *x*-intercepts of a graph, and factors of a polynomial expression)
- AII/T.9 (collect and analyze data, determine the equation of the curve of best fit, make
- predictions, and solve real-world problems, using mathematical models. Mathematical models will include polynomial, exponential, and logarithmic functions)
- AFDA.1 (investigate and analyze linear, quadratic, exponential, and logarithmic families and their characteristics)
- AFDA.2 (write an equation, given the graph of a linear, quadratic, exponential, or logarithmic function)
- AFDA.3 (collect data and generate an equation for the curve of best fit to model real-world problems or applications)

NCTM Standards

- generalize patterns using explicitly defined and recursively defined functions;
- understand relations and functions and select, convert flexibly among, and use various representations for them;
- analyze functions of one variable by investigating rates of change, intercepts, zeros, asymptotes, and local and global behavior;
- use symbolic algebra to represent and explain mathematical relationships
- identify essential quantitative relationships in a situation and determine the class or classes of functions that might model the relationships;
- use symbolic expressions, including iterative and recursive forms, to represent relationships arising from various contexts;
- draw reasonable conclusions about a situation being modeled.

Additional Objectives for Student Learning
Students will

- Use internet searches and/or websites provided to gain information regarding the significance of the problem of focus and to retrieve data
- Use technology including TI-84+ graphing calculators and Microsoft Excel to find the curve of best fit for a set of data

Materials/Resources
Classroom set of TI-84+ Graphing Calculators
Access to computer lab with internet access
http://www.treasurydirect.gov/govt/reports/pd/histdebt/histdebt.htm
http://www.washingtontimes.com/news/2007/oct/28/the-debt-bomb/
http://www.wisegeek.com/what-is-the-relationship-between-public-debt-and-gdp.htm
SmartBoard with TI Smartview Calculator or Emulator
Graph paper
Paper

Assumption of Prior Knowledge

- The typical student would have successfully completed algebra 1, algebra 2, geometry, and algebra functions with data analysis. Students should be able to write the equation of a line using two points on the line, factor polynomials, convert between graphic and symbolic forms of functions, be able to find the maximum or minimum of a function, and be able to determine the equation of the curve of best fit of a set of data, make predictions, and solve real-world problems, using mathematical models including polynomial, exponential, and logarithmic functions.
- The student has a working knowledge of Excel spreadsheets.
- To be successful with this lesson a typical student should be operating on level 2 of the Van Hiele scale – abstraction. Students have a good understanding of properties and understand that one set of properties may imply another property.
- Students may find it difficult to develop models by hand for polynomial functions of degree greater than one, and may find it difficult to develop models for data fitting an exponential or logarithmic pattern.

Concepts relevant to this unit that should be covered prior to this unit include:

- an investigation of linear, quadratic, exponential, and logarithmic families algebraically and graphically
- writing an equation given the graph of a function (linear, quadratic, exponential, and logarithmic)
- an investigation and description of the relationships among solutions of an equation, zeros of a function, x-intercepts of a graph, and factors of a polynomial expression
- an ability to recognize the general shape of a function (polynomial, exponential, and logarithmic)

Introduction: Setting Up the Mathematical Task

In this lesson, students will find a function that models a set of data and use the model to interpolate and extrapolate, i.e. make predictions of the data.

The United States public debt is a measure of the obligations of the US federal government which includes intragovernmental holdings and debt held by the public. The gross domestic product (GDP) refers to the market value of all goods and services produced within the US and is often considered an indicator of the standard of living in the US. The relationship between the national debt and GDP is one of the most important economic ratios. A declining debt/GDP ratio generally means the economy is growing faster than the national debt whereas a rising debt/GDP ratio means that the national debt is growing faster than the economy resulting in an increased burden of the national debt. Debate about raising the debt ceiling has been a news item in the recent months.

Retrieve data for both the national debt and the GDP for the years 1950 – 2010. Create a model (polynomial, exponential, or logarithmic) that represents the debt/GDP ratio over time.

- State your simplifying assumptions. (small groups 10 – 15 minutes followed by whole class 5 minutes)
- Identify dependent and independent variables in your mathematical model. (small groups 5 minutes followed by whole class 2 minutes)
- Identify other information necessary (to be gathered using technology) in order to solve the problem.

The teacher will facilitate the activity by observing and answering questions when needed. A discussion including some of the possible simplifying assumptions would be helpful. These may include:

- Assume that the national debt and GDP will not have any sudden changes.

Students will be paired in groups of three based on interests and ability levels as determined previously by teacher through surveys, observations, and assessments.

Student Exploration 1:

Students will collect data on the national debt and the GDP for the given time period. They will use prior knowledge of plotting points, estimation, and calculating the regression equation using a graphing utility to find a model for each set of data that was collected. To develop a relationship between the two sets of data over time, the national debt function $d(x)$ will be divided by the GDP function $g(x)$ where x represents time with the year 1950 as $x = 0$.

Student/Teacher Actions:

To explore and develop this model the following are important points of discussion:

- Students will enter each set of data into the Lists menu of the graphing calculator where L_1 is the year (x), L_2 is the national debt (y), and L_3 is the GDP.
- Students will adjust the viewing window of the calculator to accommodate the data that was entered
- Using the statplots menu, students will plot the data on the graphing calculator.

- Determine if the points appear to follow a linear, quadratic, exponential, or logarithmic regression
- Use the graphing calculator to find the regression equation for the data
- Using the diagnostics option on the graphing calculator, determine the correlation coefficient for the regression equation that you found
- Use Excel to graph data and determine the curve of best fit, i.e. trend (linest), exponential (growth), or logarithmic (log)
- Use the CORREL function in Excel to find the correlation coefficient
- Based on the correlation coefficient, discuss the validity of the regression equation
- What are the limitations of the models?
- Find a model to represent the ratio of debt/GDP $R(x) = \frac{d(x)}{g(x)}$
- What type of model does this ratio represent?
- Discuss the significance of this model.

Assessment

- Using the appropriate model, students will determine what the national debt will be for the years
  a) 2011
  b) 2015
  c) 2020
- Using the appropriate model, students will determine what the GDP will be for the years
  a) 2011
  b) 2015
  c) 2020
- Using the appropriate model, students will determine what the debt/GDP ratio will be for the years
  a) 2011
  b) 2015
  c) 2020

Extensions and Connections (for all students)

To extend the material for students ready to move forward have them create functions for the national debt, GDP, and the debt/GDP ratio for China.

Strategies for Differentiation

Use different tasks to find the regression equation. Students grouped in pairs according to ability levels previously determined.

- Task 1: Plot the data on graph paper using a separate graph for time vs. national debt and a separate graph for time vs. GDP. Use time as the independent variable and the national debt or GDP as the dependent variable. Estimate the regression equation using the "eyeball" method, draw the curve, and use the graphs to determine the national debt and GDP in the year 2011.
- Task 2: Plot the data on graph paper as described above. Estimate the regression equation using the "eyeball" method, and draw the curve. Identify two points on the curve and use

the points to write the equation of the curve. Use the equation to determine the national debt and the GDP in the year 2011.

- Task 3: Enter the data into the Lists menu of the graphing calculator using time as the independent variable and the national debt or GDP as the dependent variable. Use the stat plots menu to graph the points. Calculate the regression equation. Transfer this equation to the $y =$ menu and graph, adjusting the window to view the curve. Find the value of y (the national debt or GDP) when $x = 61$ (time). Find the value of the correlation coefficient and discuss its significance and affect on the prediction.
- Task 4: Enter the data into an Excel spreadsheet using time as the independent variable and the number of big game hunters or deer kill as the dependent variable. Graph the data by inserting a scatter plot. After observing the results of the graph, determine whether to use trend, exponential, or a logarithmic function to represent the data. Find the value of y (the number of big game hunters or deer kill) when $x = 11$ (time). Find the value of the correlation coefficient using CORREL and discuss its significance and affect on the prediction.

Student Exploration 2

Individual work

Students should the national debt and the consequences of raising the debt ceiling. Make a list of any websites from which you gather information.

Small Group Work

- Students will share information found individually and come to a consensus on the consequences of raising the debt ceiling.
- Students will prepare a PowerPoint presentation to include all mathematical computations calculated in Student Exploration 1, beginning with raw data, graphs, modeling functions, etc. The PowerPoint should also include their research as stated above.
- Requirements for PowerPoint:
    - A minimum of 10 slides
    - Use of graphics to display data
    - Calculations typed using the equation editor
    - Use of a common theme and design
    - A list of sources