Mathematical Modeling

I. UNIT OVERVIEW & PURPOSE:

Students will gain a deeper understanding of the use of polynomial, exponential, and logarithmic functions by applying them to real-world situations including analyzing factors that contribute to automobile accidents in Virginia caused by deer, analyzing road speed and fuel economy, and analyzing the affects of raising the US debt ceiling. They will understand the meaning of the important features of the graphs of these functions, i.e. the intercepts, maximum and/or minimum points, and the asymptotes, and make interpretations in the context of the problems.

II. UNIT AUTHOR:

Cynthia Cowley. Piedmont Governor's School.

III. COURSE:

Mathematical Modeling: Capstone Course

IV. CONTENT STRAND:

Data Analysis and Probability

V. OBJECTIVES:

Students will:

- Plot data points on graph paper to determine the type of relation (polynomial, exponential, logarithmic);
- Determine the line or curve of best fit by hand;
- Enter data into List function of graphing calculator and calculate the curve of best fit using the graphing calculator;
- Use Excel to graph data and determine the curve of best fit, i.e. trend (linest), exponential (growth), or logarithmic (log);
- Analyze the correlation coefficient to determine if the curve is a good model of the data;
- Verbalize the meaning of the pertinent information of the curve, i.e. intercepts, maximum and/or minimum points, and asymptotes in the context of the problem;
- Prepare a PowerPoint presentation to illustrate the problem, the calculations, and the discussion regarding the model found for the problem.

VI. MATHEMATICS PERFORMANCE EXPECTATION(S):

MPE.2 Collect and analyze data, determine the equation of the curve of best fit, make predictions, and solve real-world problems, using mathematical models.

Mathematical models will include polynomial, exponential, and logarithmic functions.

VII. CONTENT:

This unit will provide students with an understanding and appreciation of polynomial, exponential, and logarithmic functions by modeling real-world situations to which they can relate. Students will learn how to analyze data and create a model for the data both by hand and by using a graphing calculator. Students will also determine the limitations (if any) of such models and why they exist.

VIII. REFERENCE/RESOURCE MATERIALS:

Students will need access to a computer lab with internet access for the purpose of collecting data and completing research. A classroom set of TI-84+ graphing calculators will also be provided.

IX. PRIMARY ASSESSMENT STRATEGIES:

Students will be assessed on their process of completing the lessons including the correctness of the mathematical computations, the ability to discuss and describe the significance and the meaning of the models they calculate, and the preparation and presentation of the calculations and their significance.

X. EVALUATION CRITERIA:

Two grading rubrics are provided with lesson 1. One will be used to assess the mathematical computations and discussion and the second will be used to assess the PowerPoint and presentation. Since lesson 2 and lesson 3 are similar in nature, the same grading rubrics will be used for all lessons.

XI. INSTRUCTIONAL TIME:

Three lessons are included which are similar in nature. Teachers may wish to offer students an option of choosing one of the three to complete, or completing all three. Approximately two weeks based on a 90-minute class period would be needed if all three lessons are to be completed.

Lesson 2

Strand
Data Analysis and Probability

Mathematical Objectives
Curve of best fit. In this lesson students will collect and analyze data (websites provided under Materials and Resources) to determine the optimum speed limit for interstate travel to obtain maximum gas mileage. They will use their knowledge of polynomial, exponential, and logarithmic functions to create a mathematical model that will represent the relationship between these two variables. They will list simplifying assumptions, discuss the validity of their model by examining the correlation coefficient and discuss how the model changes if assumptions change.

Mathematics Performance Expectations

2. The student will collect and analyze data, determine the equation of the curve of best fit, make predictions, and solve real-world problems, using mathematical models. Mathematical models will include polynomial, exponential, and logarithmic functions.

Specifically, the student will:

- Recognize the general shape of a function (polynomial, exponential, and logarithmic)
- Use graphing calculator to investigate the shapes and behaviors of these functions
- Write an equation given the graph of a function
- Analyze functions to find the real-world meaning of the x- and y-intercepts, local and absolute maxima and minima, and asymptotes
- Find the value of a function for an element in its domain

Related SOL

- A.2c (perform operations on polynomials including factoring)
- A.7b,c,d (investigate and analyze function families and their characteristics both analytically and graphically including domain and range, zeros, x- and y-intercepts)
- A.11 (determine the equation of the curve of best fit)
- AII/T.8 (investigate and describe the relationships among solutions of an equation,
- zeros of a function, *x*-intercepts of a graph, and factors of a polynomial expression)
- AII/T.9 (collect and analyze data, determine the equation of the curve of best fit, make
- predictions, and solve real-world problems, using mathematical models. Mathematical models will include polynomial, exponential, and logarithmic functions)
- AFDA.1 (investigate and analyze linear, quadratic, exponential, and logarithmic families and their characteristics)
- AFDA.2 (write an equation, given the graph of a linear, quadratic, exponential, or logarithmic function)
- AFDA.3 (collect data and generate an equation for the curve of best fit to model real-world problems or applications)

NCTM Standards

- generalize patterns using explicitly defined and recursively defined functions;
- understand relations and functions and select, convert flexibly among, and use various representations for them;
- analyze functions of one variable by investigating rates of change, intercepts, zeros, asymptotes, and local and global behavior;
- use symbolic algebra to represent and explain mathematical relationships
- identify essential quantitative relationships in a situation and determine the class or classes of functions that might model the relationships;
- use symbolic expressions, including iterative and recursive forms, to represent relationships arising from various contexts;
- draw reasonable conclusions about a situation being modeled.

Additional Objectives for Student Learning

Students will

- Use internet searches and/or websites provided to gain information regarding the significance of the problem of focus and to retrieve data
- Use technology including TI-84+ graphing calculators and Microsoft Excel to find the curve of best fit for a set of data

Materials/Resources

Classroom set of TI-84+ Graphing Calculators
Access to computer lab with internet access
Possible websites to gather data and information
http://www.metrompg.com/posts/speed-vs-mpg.htm
*http://cta.ornl.gov/data/tedb29/Edition29_Chapter04.pdf
http://www.fueleconomy.gov/feg/drivehabits.shtml
SmartBoard with TI Smartview Calculator or Emulator
Graph paper
Paper

Assumption of Prior Knowledge

- The typical student would have successfully completed algebra 1, algebra 2, geometry, and algebra functions with data analysis. Students should be able to write the equation of a line using two points on the line, factor polynomials, convert between graphic and symbolic forms of functions, be able to find the average of a set of data, be able to find the maximum or minimum of a function, and be able to determine the equation of the curve of best fit of a set of data, make predictions, and solve real-world problems, using mathematical models including polynomial, exponential, and logarithmic functions.
- To be successful with this lesson a typical student should be operating on level 2 of the Van Hiele scale – abstraction. Students have a good understanding of properties and understand that one set of properties may imply another property.
- Students may find it difficult to develop models by hand for polynomial functions of degree greater than one, and may find it difficult to develop models for data fitting an exponential or logarithmic pattern.

Concepts relevant to this unit that should be covered prior to this unit include:

- an investigation of linear, quadratic, exponential, and logarithmic families algebraically and graphically
- writing an equation given the graph of a function (linear, quadratic, exponential, and logarithmic)
- an investigation and description of the relationships among solutions of an equation, zeros of a function, x-intercepts of a graph, and factors of a polynomial expression
- an ability to recognize the general shape of a function (polynomial, exponential, and logarithmic)

Introduction: Setting Up the Mathematical Task

In this lesson, students will find a function that models a set of data and use the model to interpolate and extrapolate, i.e. make predictions of the data.

The primary focus of fuel conservation is typically on the fuel economy of new cars and light trucks, but it is widely recognized that vehicle owners can take steps to maximize fuel economy of their vehicle in-use. Setting the maximum speed limit on interstates is often an issue that arises in times of higher gas prices. Collect and analyze data to determine the optimum speed to maximize fuel economy on interstate driving.

- State your simplifying assumptions. (small groups 10 – 15 minutes followed by whole class 5 minutes)
- Identify dependent and independent variables in your mathematical model. (small groups 5 minutes followed by whole class 2 minutes)
- Identify other information necessary (to be gathered using technology) in order to solve the problem.

The teacher will facilitate the activity by observing and answering questions when needed. A discussion including some of the possible simplifying assumptions would be helpful. These may include:

1. Vehicles considered will be the most popular passenger cars used in America in 1997*.
2. Motorists will be responsible drivers, not of the aggressive nature.
3. Vehicle owners have routine maintenance performed on their vehicles.
4. Vehicle owners use tires that have low rolling resistance, thus helping to improve gas mileage.
5. Vehicle owners use fuel efficient lubricants.

Students will be paired in groups of three based on interests and ability levels as determined previously by teacher through surveys, observations, and assessments.

Student Exploration 1:

Students will collect data on maximum fuel efficiency for the most popular U.S. models in 1997*. They will use prior knowledge of plotting points, estimation, and calculating the regression equation using a graphing utility to find a model for the data that was collected.

Student/Teacher Actions:

To explore and develop this model the following are important points of discussion:

- Students will research the fuel efficiency of each of the models in the 1997 study. Data needed includes the mpg ratio at speeds ranging from 30 to 75 mph.
- Students will find the average fuel efficiency of the models selected.
- Students will enter each set of data into the Lists menu of the graphing calculator where L_1 is speed (x), and L_2 is average fuel efficiency (y).
- Students will adjust the viewing window of the calculator to accommodate the data that was entered
- Using the statplots menu, students will plot the data on the graphing calculator.
- Determine if the points appear to follow a linear, quadratic, exponential, or logarithmic regression
- Use the graphing calculator to find the regression equation for the data
- Using the diagnostics option on the graphing calculator, determine the correlation coefficient for the regression equation that you found
- Use Excel to graph data and determine the curve of best fit, i.e. trend (linest), exponential (growth), or logarithmic (log).
- Use the CORREL function in Excel to find the correlation coefficient.
- Compare graph and answers to those found with the graphing calculator.
- Based on the correlation coefficient, discuss the validity of the regression equation
- What are the limitations of the model?
- Discuss the affect of speed on average fuel efficiency.

Assessment

- Using the appropriate model, students will determine what the fuel economy for speeds
  a) 62 miles per hour
  b) 82 miles per hour
  c) 85 miles per hour
- For what speed will the gas mileage be 32 mpg?

Extensions and Connections (for all students)

To extend the material for students ready to move forward have them gather data needed to update the study completed in 1997.

Strategies for Differentiation

Use different tasks to find the regression equation. Students grouped in pairs according to ability levels previously determined.

- Task 1: Plot the data on graph paper using the speed as the independent variable and the average fuel efficiency as the dependent variable, estimate the regression equation using the “eyeball” method, draw the curve, and use the graph to determine the maximum fuel economy.
- Task 2: Plot the data on graph paper using the speed as the independent variable and the average fuel efficiency as the dependent variable, estimate the regression equation using

the "eyeball" method, and draw the curve. Identify two points on the curve, set up a system of equations to find the equation of the curve. Use the equation to determine the maximum fuel economy.

- Task 3: Enter the data into the Lists menu of the graphing calculator using the speed as the independent variable and the average fuel efficiency as the dependent variable. Use the stat plots menu to graph the points. Calculate the regression equation. Transfer this equation to the $y =$ menu and graph, adjusting the window to view the curve. Calculate the maximum point on the curve using the calc menu. Find the value of y (the average fuel efficiency) of the maximum point. Note that x represents the speed that yields the average fuel efficiency. Find the value of the correlation coefficient and discuss its significance and affect on the prediction.
- Task 4: Enter the data into an Excel spreadsheet using speed as the independent variable and the average fuel efficiency as the dependent variable. Graph the data by inserting a scatter plot. After observing the results of the graph, determine whether to use trend, exponential, or a logarithmic function to represent the data. Find the value of y (the average fuel efficiency) at its maximum point. Find the value of the correlation coefficient using CORREL and discuss its significance and affect on the prediction.

Lesson 3

Strand
Data Analysis and Probability

Mathematical Objectives
Curve of best fit. In this lesson students will collect and analyze data (websites provided under Materials and Resources) to determine a function to represent the national debt. They will use their knowledge of polynomial, exponential, and logarithmic functions to create a mathematical model that will represent the relationship between these two variables. They will list simplifying assumptions, discuss the validity of their model by examining the correlation coefficient and discuss how the model changes if assumptions change.

Mathematics Performance Expectations

2. The student will collect and analyze data, determine the equation of the curve of best fit, make predictions, and solve real-world problems, using mathematical models. Mathematical models will include polynomial, exponential, and logarithmic functions.

Specifically, the student will:

- Recognize the general shape of a function (polynomial, exponential, and logarithmic)
- Use graphing calculator to investigate the shapes and behaviors of these functions
- Write an equation given the graph of a function
- Analyze functions to find the real-world meaning of the x- and y-intercepts, local and absolute maxima and minima, and asymptotes
- Find the value of a function for an element in its domain

Related SOL

- A.2c (perform operations on polynomials including factoring)
- A.7b,c,d (investigate and analyze function families and their characteristics both analytically and graphically including domain and range, zeros, x- and y-intercepts)
- A.11 (determine the equation of the curve of best fit)
- AII/T.8 (investigate and describe the relationships among solutions of an equation,
- zeros of a function, *x*-intercepts of a graph, and factors of a polynomial expression)
- AII/T.9 (collect and analyze data, determine the equation of the curve of best fit, make
- predictions, and solve real-world problems, using mathematical models. Mathematical models will include polynomial, exponential, and logarithmic functions)
- AFDA.1 (investigate and analyze linear, quadratic, exponential, and logarithmic families and their characteristics)
- AFDA.2 (write an equation, given the graph of a linear, quadratic, exponential, or logarithmic function)
- AFDA.3 (collect data and generate an equation for the curve of best fit to model real-world problems or applications)

NCTM Standards

- generalize patterns using explicitly defined and recursively defined functions;
- understand relations and functions and select, convert flexibly among, and use various representations for them;
- analyze functions of one variable by investigating rates of change, intercepts, zeros, asymptotes, and local and global behavior;
- use symbolic algebra to represent and explain mathematical relationships
- identify essential quantitative relationships in a situation and determine the class or classes of functions that might model the relationships;
- use symbolic expressions, including iterative and recursive forms, to represent relationships arising from various contexts;
- draw reasonable conclusions about a situation being modeled.

Additional Objectives for Student Learning

Students will

- Use internet searches and/or websites provided to gain information regarding the significance of the problem of focus and to retrieve data
- Use technology including TI-84+ graphing calculators and Microsoft Excel to find the curve of best fit for a set of data

Materials/Resources

Classroom set of TI-84+ Graphing Calculators
Access to computer lab with internet access
http://www.treasurydirect.gov/govt/reports/pd/histdebt/histdebt.htm
http://www.washingtontimes.com/news/2007/oct/28/the-debt-bomb/
http://www.wisegeek.com/what-is-the-relationship-between-public-debt-and-gdp.htm
SmartBoard with TI Smartview Calculator or Emulator
Graph paper

Paper

Assumption of Prior Knowledge

- The typical student would have successfully completed algebra 1, algebra 2, geometry, and algebra functions with data analysis. Students should be able to write the equation of a line using two points on the line, factor polynomials, convert between graphic and symbolic forms of functions, be able to find the maximum or minimum of a function, and be able to determine the equation of the curve of best fit of a set of data, make predictions, and solve real-world problems, using mathematical models including polynomial, exponential, and logarithmic functions.
- The student has a working knowledge of Excel spreadsheets.
- To be successful with this lesson a typical student should be operating on level 2 of the Van Hiele scale – abstraction. Students have a good understanding of properties and understand that one set of properties may imply another property.
- Students may find it difficult to develop models by hand for polynomial functions of degree greater than one, and may find it difficult to develop models for data fitting an exponential or logarithmic pattern.

Concepts relevant to this unit that should be covered prior to this unit include:

- an investigation of linear, quadratic, exponential, and logarithmic families algebraically and graphically
- writing an equation given the graph of a function (linear, quadratic, exponential, and logarithmic)
- an investigation and description of the relationships among solutions of an equation, zeros of a function, x-intercepts of a graph, and factors of a polynomial expression
- an ability to recognize the general shape of a function (polynomial, exponential, and logarithmic)

Introduction: Setting Up the Mathematical Task

In this lesson, students will find a function that models a set of data and use the model to interpolate and extrapolate, i.e. make predictions of the data.

The United States public debt is a measure of the obligations of the US federal government which includes intragovernmental holdings and debt held by the public. The gross domestic product (GDP) refers to the market value of all goods and services produced within the US and is often considered an indicator of the standard of living in the US. The relationship between the national debt and GDP is one of the most important economic ratios. A declining debt/GDP ratio generally means the economy is growing faster than the national debt whereas a rising debt/GDP ratio means that the national debt is growing faster than the economy resulting in an increased burden of the national debt. Debate about raising the debt ceiling has been a news item in the recent months.

Retrieve data for both the national debt and the GDP for the years 1950 – 2010. Create a model (polynomial, exponential, or logarithmic) that represents the debt/GDP ratio over time.

- State your simplifying assumptions. (small groups 10 – 15 minutes followed by whole class 5 minutes)
- Identify dependent and independent variables in your mathematical model. (small groups 5 minutes followed by whole class 2 minutes)
- Identify other information necessary (to be gathered using technology) in order to solve the problem.

The teacher will facilitate the activity by observing and answering questions when needed. A discussion including some of the possible simplifying assumptions would be helpful. These may include:

- Assume that the national debt and GDP will not have any sudden changes.

Students will be paired in groups of three based on interests and ability levels as determined previously by teacher through surveys, observations, and assessments.

Student Exploration 1:

Students will collect data on the national debt and the GDP for the given time period. They will use prior knowledge of plotting points, estimation, and calculating the regression equation using a graphing utility to find a model for each set of data that was collected. To develop a relationship between the two sets of data over time, the national debt function $d(x)$ will be divided by the GDP function $g(x)$ where x represents time with the year 1950 as $x = 0$.

Student/Teacher Actions:

To explore and develop this model the following are important points of discussion:

- Students will enter each set of data into the Lists menu of the graphing calculator where L_1 is the year (x), L_2 is the national debt (y), and L_3 is the GDP.
- Students will adjust the viewing window of the calculator to accommodate the data that was entered
- Using the statplots menu, students will plot the data on the graphing calculator.
- Determine if the points appear to follow a linear, quadratic, exponential, or logarithmic regression
- Use the graphing calculator to find the regression equation for the data
- Using the diagnostics option on the graphing calculator, determine the correlation coefficient for the regression equation that you found
- Use Excel to graph data and determine the curve of best fit, i.e. trend (linest), exponential (growth), or logarithmic (log)
- Use the CORREL function in Excel to find the correlation coefficient
- Based on the correlation coefficient, discuss the validity of the regression equation
- What are the limitations of the models?
- Find a model to represent the ratio of debt/GDP $R(x) = \frac{d(x)}{g(x)}$
- What type of model does this ratio represent?
- Discuss the significance of this model.

Assessment

- Using the appropriate model, students will determine what the national debt will be for the years
  a) 2011
  b) 2015
  c) 2020
- Using the appropriate model, students will determine what the GDP will be for the years
  a) 2011
  b) 2015
  c) 2020
- Using the appropriate model, students will determine what the debt/GDP ratio will be for the years
  a) 2011
  b) 2015
  c) 2020

Extensions and Connections (for all students)

To extend the material for students ready to move forward have them create functions for the national debt, GDP, and the debt/GDP ratio for China.

Strategies for Differentiation

Use different tasks to find the regression equation. Students grouped in pairs according to ability levels previously determined.

- Task 1: Plot the data on graph paper using a separate graph for time vs. national debt and a separate graph for time vs. GDP. Use time as the independent variable and the national debt or GDP as the dependent variable. Estimate the regression equation using the "eyeball" method, draw the curve, and use the graphs to determine the national debt and GDP in the year 2011.
- Task 2: Plot the data on graph paper as described above. Estimate the regression equation using the "eyeball" method, and draw the curve. Identify two points on the curve and use the points to write the equation of the curve. Use the equation to determine the national debt and the GDP in the year 2011.
- Task 3: Enter the data into the Lists menu of the graphing calculator using time as the independent variable and the national debt or GDP as the dependent variable. Use the stat plots menu to graph the points. Calculate the regression equation. Transfer this equation to the $y =$ menu and graph, adjusting the window to view the curve. Find the value of y (the national debt or GDP) when $x = 61$ (time). Find the value of the correlation coefficient and discuss its significance and affect on the prediction.
- Task 4: Enter the data into an Excel spreadsheet using time as the independent variable and the number of big game hunters or deer kill as the dependent variable. Graph the data by inserting a scatter plot. After observing the results of the graph, determine whether to use trend, exponential, or a logarithmic function to represent the data. Find the value of y (the number of big game hunters or deer kill) when $x = 11$ (time). Find the value of the correlation coefficient using CORREL and discuss its significance and affect on the prediction.

Student Exploration 2

Individual work

Students should the national debt and the consequences of raising the debt ceiling. Make a list of any websites from which you gather information.

Small Group Work

- Students will share information found individually and come to a consensus on the consequences of raising the debt ceiling.
- Students will prepare a PowerPoint presentation to include all mathematical computations calculated in Student Exploration 1, beginning with raw data, graphs, modeling functions, etc. The PowerPoint should also include their research as stated above.
- Requirements for PowerPoint:
    - A minimum of 10 slides
    - Use of graphics to display data
    - Calculations typed using the equation editor
    - Use of a common theme and design
    - A list of sources